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in front of the single-layer and double-layer potentials is replaced by $1/(2\alpha)$, where α is the angle subtended by the corner.

2.6.5 Derive the fundamental solution to the biharmonic equation (2.6.28).

2.6.6 Using the second Green's identity derive the boundary integral equation (2.6.32).

2.6.7 Verify that the Green's functions (2.6.29) and (2.6.30) satisfy the boundary conditions $G = 0$ and $\nabla G \cdot \mathbf{n} = 0$ over the corresponding bounding surfaces.

2.6.8 Show that $\nabla \omega \cdot \mathbf{n} = -(1/\mu)\nabla P \cdot \mathbf{t}$, where $\mathbf{n}$ and $\mathbf{t}$ are the unit normal and unit tangent vectors with respect to a line in a flow.

2.6.9 Verify that (2.6.35) provides us with two solutions of the biharmonic equation.

2.7 Unsteady Stokes flow

Turning now our attention to unsteady Stokes flow, we consider the time-transformed non-dimensional unsteady Stokes equation (1.6.9), and repeat our previous analysis for steady flow. Recall that, for convenience, we dropped the primes indicating non-dimensional variables and incorporated the effect of the body force into the modified pressure. Under these simplifications, we find that the Green's functions of unsteady Stokes flow represent solutions of the continuity equation and the singularly forced unsteady Stokes equation

$$(\nabla^2 - \lambda^2)\mathbf{u} = \nabla P - \mathbf{g}\delta(\mathbf{x} - \mathbf{x}_0) \tag{2.7.1}$$

where $\mathbf{g}$ is a constant vector.

The unsteady Stokeslet

To compute the free-space Green's function, we follow a procedure similar to that outlined in section 2.2. Thus, we substitute (2.2.1), (2.2.2), and (2.2.4) into (2.7.1) to obtain

$$(\nabla^2 - \lambda^2)H = -\frac{1}{4\pi r} \tag{2.7.2}$$

where $r = |\hat{\mathbf{x}}|$, $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$. Using (2.2.1) we find that H is the fundamental solution of the modified biharmonic equation $\nabla^2(\nabla^2 - \lambda^2)H = \delta(\mathbf{x} - \mathbf{x}_0)$:

$$H = \frac{1}{4\pi\lambda R}(1 - e^{-R}) \tag{2.7.3}$$

where $R = \lambda r$. Substituting (2.7.3) into (2.2.4) we derive the velocity field due to an unsteady point force in the standard form

$$u_i(\mathbf{x}, \mathbf{x}_0) = \frac{1}{8\pi}\mathcal{S}_{ij}(\hat{\mathbf{x}})g_j \tag{2.7.4}$$

where

$$\mathcal{S}_{ij}(\hat{\mathbf{x}}) = A(R)\frac{\delta_{ij}}{r} + B(R)\frac{\hat{x}_i\hat{x}_j}{r^3} \tag{2.7.5}$$

is the *unsteady Stokeslet*. The functions $A(R)$ and $B(R)$ are defined as

$$A = 2e^{-R}\left(1 + \frac{1}{R} + \frac{1}{R^2}\right) - \frac{2}{R^2} \qquad B = -2e^{-R}\left(1 + \frac{3}{R} + \frac{3}{R^2}\right) + \frac{6}{R^2} \tag{2.7.6}$$

It will be noted that $A(0) = B(0) = 1$, suggesting that at small frequencies or close to the pole, the unsteady Stokeslet reduces to the regular Stokeslet for steady flow. To examine the asymptotic limit of small frequencies in more detail, we expand $\mathcal{S}$ in a Taylor series for small λ obtaining

$$\mathcal{S} = \mathcal{S}^0 + \lambda\mathcal{S}^1 + \lambda^2\mathcal{S}^2 + \lambda^3\mathcal{S}^3 \cdots \tag{2.7.7}$$

where $\mathcal{S}^0$ is the steady Stokeslet, and

$$\mathcal{S}^1_{ij} = -\tfrac{4}{3}\delta_{ij} \qquad \mathcal{S}^2_{ij} = \frac{1}{4}\left(3r\delta_{ij} - \frac{\hat{x}_i\hat{x}_j}{r}\right) \qquad \mathcal{S}^3_{ij} = \tfrac{2}{15}(2r^2\delta_{ij} - \hat{x}_i\hat{x}_j) \tag{2.7.8}$$

It will be useful to note that $\mathcal{S}^1$ represents streaming flow.

To examine the behaviour of the Green's function at high values of λ or far away from the pole, we expand $\mathcal{S}$ in an asymptotic series for large R obtaining

$$\mathcal{S}_{ij} = \frac{2}{\lambda^2}\left(-\frac{\delta_{ij}}{r^3} + \frac{\hat{x}_i\hat{x}_j}{r^5}\right) + 2e^{-R}\left(\frac{\delta_{ij}}{r} - \frac{\hat{x}_i\hat{x}_j}{r^3}\right) + \cdots \tag{2.7.9}$$

In order to ensure that $\mathcal{S}$ vanishes at infinity, we require that the real part of λ is positive. We note that the expression in the first parenthesis on the right-hand side of (2.7.9) is the steady potential dipole; this suggests that at high frequencies or large distances, the unsteady Stokeslet produces irrotational flow (see section 7.2).

The vorticity, pressure, and stress fields associated with the unsteady point force are given by

$$\omega_i = \frac{1}{8\pi}\Omega_{ij}g_j \qquad p = \frac{1}{8\pi}p_j g_j \qquad \sigma_{ik} = \frac{1}{8\pi}T_{ijk}g_j \tag{2.7.10}$$

where

$$\Omega_{ij} = 2\varepsilon_{ijl}\frac{\hat{x}_l}{r^3}e^{-R}(R + 1) \tag{2.7.11}$$

$$p_i = 2\frac{\hat{x}_i}{r^3} \tag{2.7.12}$$

and

$$T_{ijk} = -\frac{2}{r^3}(\delta_{ij}\hat{x}_k + \delta_{kj}\hat{x}_i)[e^{-R}(R+1) - B] - \frac{2}{r^3}\delta_{ik}\hat{x}_j(1-B)$$
$$-2\frac{\hat{x}_i\hat{x}_j\hat{x}_k}{r^5}[5B - 2e^{-R}(R+1)] \tag{2.7.13}$$

When the domain of flow is infinite, all $\mathbf{\Omega}$, $\mathbf{p}$, and $\mathbf{T}$ are required to decay to zero as the observation point moves far away from the pole.

The pressure vector $\mathbf{p}$ and stress tensor $\mathbf{T}$ are two acceptable unsteady Stokes flows representing a point source and a stresslet respectively (see section 2.2 and also (2.1.14) and (2.1.15)). The pressure matrix $\mathbf{\Pi}$ corresponding to the stresslet will be discussed in problem 2.7.6.

It will be instructive to consider the surface force on a spherical surface that is centered at the unsteady point force. After some algebra we find

$$f_i = \sigma_{ij}n_j = \frac{1}{8\pi}\left[\frac{\delta_{ij}}{r^2}K(R) + \frac{\hat{x}_i\hat{x}_j}{r^4}L(R)\right]g_j \tag{2.7.14}$$

where the functions $K(R)$ and $L(R)$ are defined as

$$K = 2[B - e^{-R}(R+1)] \qquad L = 2[e^{-R}(R+1) - 1 - 3B] \tag{2.7.15}$$

and the function B was given in (2.7.6). One may show that $K(0) = 0$ and $L(0) = -6$, consistent with out previous results for steady Stokes flow. Using (2.7.14) we find that the force exerted on the spherical surface is given by

$$\mathbf{F} = \tfrac{1}{6}(3K + L)\mathbf{g} = -\tfrac{1}{3}[2e^{-R}(R+1) + 1]\mathbf{g} \tag{2.7.16}$$

We note that the force acting on a small sphere of infinitesimal radius is equal to $-\mathbf{g}$, whereas the force on a sphere of infinitely large radius is equal to $-\frac{1}{3}\mathbf{g}$. The difference between these two values is equal to the rate of change of momentum of the fluid that surrounds the point force.

The boundary integral equation

To derive a boundary integral equation for unsteady Stokes flow, we follow the procedure outlined in section 2.3. In this manner, we derive (2.3.4), (2.3.11), and (2.3.13) for points in the exterior, interior, and on the boundary of the flow. The properties of these equations are similar to those of the corresponding equations for steady flow discussed in the preceding sections. It is worth noting, in particular, that the boundary integral equation for unsteady flow may be simplified by eliminating either the single-layer or the double-layer potential as discussed in section 2.3 (see problem 2.7.4). The pressure field is given by the boundary integral representation (2.3.17). The pressure matrix $\mathbf{\Pi}$ of the free-space Green's function will be discussed in problem 2.7.6.

It will prove useful to examine the asymptotic behaviour of the boundary integral equation at small values of the frequency parameter λ. Following Williams (1966b), we expand the Green's function $\mathbf{G}$ (which for simplicity we identify with the unsteady Stokeslet $\mathcal{S}$) and its associated stress tensor $\mathbf{T}$ in a Taylor series with respect to λ, as indicated in (2.7.7). Furthermore, we expand the boundary surface force $\mathbf{f}$ and the boundary velocity $\mathbf{u}$ in Taylor series with respect to λ, as $\mathbf{f} = \mathbf{f}^0 + \lambda\mathbf{f}^1 + \ldots$, $\mathbf{u} = \mathbf{u}^0 + \lambda\mathbf{u}^1 + \lambda\mathbf{u}^2 + \ldots$ Substituting these expansions into (2.3.13) and collecting terms of zero, first, and second order in λ we obtain

$$u_j^0(\mathbf{x}_0) = -\frac{1}{4\pi}\int_D f_i^0(\mathbf{x})\mathcal{S}_{ij}^0(\mathbf{x},\mathbf{x}_0)\,dS(\mathbf{x}) + \frac{1}{4\pi}\int_D^{\mathcal{PV}} u_i^0(\mathbf{x})T_{ijk}^0(\mathbf{x},\mathbf{x}_0)n_k(\mathbf{x})\,dS(\mathbf{x}) \tag{2.7.17}$$

$$u_j^1(\mathbf{x}_0) - \frac{1}{3\pi}F_j^0 = -\frac{1}{4\pi}\int_D f_i^1(\mathbf{x})\mathcal{S}_{ij}^0(\mathbf{x},\mathbf{x}_0)\,dS(\mathbf{x}) + \frac{1}{4\pi}\int_D^{\mathcal{PV}} u_i^1(\mathbf{x})T_{ijk}^0(\mathbf{x},\mathbf{x}_0)n_k(\mathbf{x})\,dS(\mathbf{x}) \tag{2.7.18}$$

and

$$u_j^2(\mathbf{x}_0) - \frac{1}{3\pi}F_j^1 = -\frac{1}{4\pi}\int_D f_i^2(\mathbf{x})\mathcal{S}_{ij}^0(\mathbf{x},\mathbf{x}_0)\,dS(\mathbf{x}) - \frac{1}{4\pi}\int_D f_i^0(\mathbf{x})\mathcal{S}_{ij}^2(\mathbf{x},\mathbf{x}_0)\,dS(\mathbf{x})$$
$$+\frac{1}{4\pi}\int_D^{\mathcal{PV}} u_i^2(\mathbf{x})T_{ijk}^0(\mathbf{x},\mathbf{x}_0)n_k(\mathbf{x})\,dS(\mathbf{x})$$
$$+\frac{1}{4\pi}\int_D u_i^0(\mathbf{x})T_{ijk}^2(\mathbf{x},\mathbf{x}_0)n_k(\mathbf{x})\,dS(\mathbf{x}) \tag{2.7.19}$$

where

$$\mathbf{F}^\alpha = \int_D \mathbf{f}^\alpha\,dS \qquad \alpha = 0, 1, 2\ldots \tag{2.7.20}$$

is the force acting on the boundary D. Prescribing the boundary velocity renders (2.7.17), (2.7.18), and (2.7.19) a system of Fredholm integral equations of the first kind for the boundary surface forces $\mathbf{f}^0$, $\mathbf{f}^1$, and $\mathbf{f}^2$.

As a specific application of the above equations, we consider the flow due to the translational or rotational vibrations of a rigid particle in an infinite fluid. Neglecting the integrals at infinity and imposing the boundary conditions $\mathbf{u}^0 = \mathbf{V} + \mathbf{\Omega}\times\mathbf{x}$ and $\mathbf{u}^1 = \mathbf{u}^2 = \cdots = 0$ on the surface of the particle, where $\mathbf{V}$ and $\mathbf{\Omega}$ are the translational and rotational velocities of oscillation, we obtain two integral equations for $\mathbf{f}^0$ and $\mathbf{f}^1$, namely

$$V_j + \varepsilon_{jkl}\Omega_k x_l = -\frac{1}{8\pi}\int_{S_m} f_i^0(\mathbf{x})\mathcal{S}_{ij}^0(\mathbf{x},\mathbf{x}_0)\,dS(\mathbf{x}) \tag{2.7.21}$$

and

$$-\frac{1}{6\pi}F_j^0 = -\frac{1}{8\pi}\int_{S_P} f_i^1(\mathbf{x})\mathcal{S}_{ij}^0(\mathbf{x},\mathbf{x}_0)\,\mathrm{d}S(\mathbf{x}) \tag{2.7.22}$$

where S_P is the surface of the particle. It will be convenient to express the solution of (2.7.21) in the form

$$\mathbf{f}^0 = -\mathcal{G}^T\cdot\mathbf{V} - \mathcal{G}^R\cdot\boldsymbol{\Omega} \tag{2.7.23}$$

where $\mathcal{G}^T$ and $\mathcal{G}^R$ are respectively the steady translational and steady rotary surface force resistance matrices introduced in (1.4.6) and (1.4.11). Using (1.4.18) and (1.4.19) we find that the steady force and steady torque exerted on the particle are given by

$$\mathbf{F}^0 = -\mathbf{X}\cdot\mathbf{V} - \mathbf{P}\cdot\boldsymbol{\Omega}, \qquad \mathbf{L}^0 = -\mathbf{P}'\cdot\mathbf{V} - \mathbf{Y}\cdot\boldsymbol{\Omega} \tag{2.7.24}$$

where $\mathbf{X}$, $\mathbf{P}$, $\mathbf{P}'$, and $\mathbf{Y}$ are the steady force and steady torque resistance matrices. Comparing the last four equations it becomes evident that the first corrections to the surface force, force, and torque are given by

$$\mathbf{f}^1 = \frac{1}{6\pi}\mathcal{G}^T\cdot\mathbf{F}^0 = -\frac{1}{6\pi}\mathcal{G}^T\cdot(\mathbf{X}\cdot\mathbf{V} + \mathbf{P}\cdot\boldsymbol{\Omega}) \tag{2.7.25}$$

$$\mathbf{F}^1 = \frac{1}{6\pi}\mathbf{X}\cdot\mathbf{F}^0 = -\frac{1}{6\pi}\mathbf{X}\cdot(\mathbf{X}\cdot\mathbf{V} + \mathbf{P}\cdot\boldsymbol{\Omega}) \tag{2.7.26}$$

and

$$\mathbf{L}^1 = \frac{1}{6\pi}\mathbf{P}'\cdot\mathbf{F}^0 = -\frac{1}{6\pi}\mathbf{P}'\cdot(\mathbf{X}\cdot\mathbf{V} + \mathbf{P}\cdot\boldsymbol{\Omega}) \tag{2.7.27}$$

Remarkably, we find that the first-order corrections may be computed directly from the resistance matrices for steady flow.

Problems

2.7.1 Verify that as R tends to zero, the vorticity and stress tensors $\boldsymbol{\Omega}$ and $\mathbf{T}$ defined in (2.7.11) and (2.7.13) reduce to those for steady flow given in (2.2.9) and (2.2.11).

2.7.2 The Laplace transform of the velocity field due to the impulsive point force $\mathbf{g}\delta(\mathbf{x}-\mathbf{x}_0)\delta(t-t_0)$ where $\mathbf{g}$ is a constant, is given by

$$\hat{u}_i(\hat{\mathbf{x}};s) = \frac{1}{8\pi}\mathcal{S}_{ij}(\hat{\mathbf{x}})g_j \tag{1}$$

where $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$ and $\mathcal{S}$ is the unsteady Stokeslet with $\lambda^2 = s$. To compute the long-time behaviour of the flow we use the expansion (2.7.7) finding

$$\hat{u}_i(\hat{\mathbf{x}};s) = \frac{1}{8\pi}[\mathcal{S}_{ij}^0(\hat{\mathbf{x}}) + s^{1/2}\mathcal{S}_{ij}^1(\hat{\mathbf{x}}) + s\mathcal{S}_{ij}^2(\hat{\mathbf{x}}) + s^{3/2}\mathcal{S}_{ij}^3(\hat{\mathbf{x}}) + \cdots]g_j \tag{2}$$

Inverting (2) show that the long-time behaviour of the flow is described by

$$u_i(\hat{\mathbf{x}};t) = \frac{1}{(4\pi t)^{3/2}}\left[-\tfrac{1}{2}\mathcal{S}_{ij}^1(\hat{\mathbf{x}}) + \frac{3}{4t}\mathcal{S}_{ij}^3(\hat{\mathbf{x}}) + O\left(\frac{r^4}{t^2}\right)\right]g_j \tag{3}$$

Note that equation (3) finds application in the computation of the long-time decay of the angular velocity autocorrelation function of a rigid Brownian particle (Hocquart & Hinch 1983).

2.7.3 Show that the generalized Faxen relations discussed in section 2.5 apply also for unsteady Stokes flow (Pozrikidis 1989a).

2.7.4 Show that an unsteady Stokes flow with a prescribed constant velocity $\mathbf{u} = \mathbf{U}$ on the surface S_P of a body may be represented in terms of a single-layer potential as

$$u_j(\mathbf{x}_0) = -\frac{1}{8\pi}\int_{S_P}[f_i(\mathbf{x}) - \lambda^2\mathbf{V}\cdot\mathbf{x}n_i(\mathbf{x})]G_{ij}(\mathbf{x},\mathbf{x}_0)\,\mathrm{d}S(\mathbf{x}) \tag{1}$$

2.7.5 The Green's function for two-dimensional unsteady flow may be derived in a procedure analogous to that described in section 2.6. The results may be expressed in terms of the fundamental solution of the two-dimensional Helmholtz equation (Stakgold 1968, Vol. II, p. 265). Derive the two-dimensional unsteady Stokeslet and show that in the limit of small λ, it reduces to the steady Stokeslet.

2.7.6 Using the results of section 7.5 show that the equivalent of (2.2.13) for unsteady flow is

$$\Pi_{ik}(\mathbf{x}_0,\mathbf{x}) = 2\frac{\delta_{ik}}{r^3}(R^2-2) + 12\frac{\hat{x}_i\hat{x}_k}{r^5}$$

2.8 Swirling flow

In section 2.4 we developed a simplified boundary integral representation for flow in an axisymmetric domain. In this section we wish to develop an even more simplified representation for swirling flow produced by the axial rotation of an axisymmetric body. An example is the flow produced by the axial rotation of a spheroid whose major axis is aligned with the centre line of a cylindrical tube. Admittedly, a swirling flow may be treated within the general framework of the boundary integral equation for flow in an axisymmetric domain. It will be useful and instructive, however, to pursue an independent derivation based on the simplified equations of flow (see section 1.2).

First, we consider steady swirling flow. In section 1.2 we saw that the swirl $\Omega = \sigma u_\phi$ satisfies the equation $E^2\Omega = 0$, where u_ϕ is the azimuthal component of the velocity and the operator E^2 was defined in (1.2.25). The boundary conditions require that on the surface of the body $u_\phi = W\sigma$ or $\Omega = W\sigma^2$, where W is the angular velocity of rotation.