1. Derive two formulas for approximating \( f'(x) \) using seven points:

(a) Use \( f(x - 3h), f(x - 2h), f(x - h), f(x), f(x + h), f(x + 2h), \) and \( f(x + 3h) \), write the approximation as a linear combination of these values and expand it in a Taylor series. Write down all equations that define the coefficients. Find the coefficients and write your formula including the leading error term. Please use Mathematica or Maple, do not do it by hand.

(b) Starting with \( N_2(h) = \frac{1}{2h}[f(x + h) - f(x - h)] \), use Richardson extrapolation twice.

2. Suppose you approximate \( f''(x) \) using the five points \( f(x - 2h), f(x - h), f(x), f(x + h) \) and \( f(x + 2h) \). Write the Taylor series expansion for each about the point \( x \). Write the linear combination
\[
Af(x - 2h) + Bf(x - h) + Cf(x) + Df(x + h) + Ef(x + 2h)
\]
and find the equations that the coefficients must satisfy. Solve for the coefficients and write the formula, including the first error term (it should contain \( h \) to some power). Choose your favorite function (it should be a good one) and evaluate its second derivative at \( x = 2 \). Use your formula to approximate \( f''(2) \) with \( h = 0.1, 0.05, 0.025 \). Write a table with the values and the errors. What is the observed convergence? does it agree with your derived error term?

3. Consider the differential equation \( y'(t) = \sin(t) \) with initial condition \( y(0) = -1 \).

(a) Find the exact solution.

(b) To find a numerical solution up to \( T = 4 \), let \( n = 40 \) and define \( h = T/n \). Now approximate the equation \( y'(t) = \sin(t) \) by
\[
\frac{y(t + h) - y(t)}{h} = \sin(t), \quad y(0) = -1
\]
and solve for \( y(t + h) \). For \( t = 0 \) you can find \( y(h) \). For \( t = h \) you can then find \( y(2h) \), and so on. This way you can find the numerical solution at \( t = 0, h, 2h, 3h, \ldots, nh \) (so the final time is \( nh = T = 4 \)). Plot the exact solution and the numerical solution in the same graph. Compute the error in your solution:
\[
Err = \text{Max}(|y(k) - y_{\text{exact}}(k)|).
\]

(c) Repeat part (b) with \( n = 80 \) (and find the numerical solution up to \( nh = T = 4 \)). Compare the results (the errors) with those of part (b). Explain what you conclude. You may want to do it for larger values of \( n \) to see the pattern.

4. Now consider the differential equation \( y'(t) = \sin(\sqrt{t} + t) \) with initial condition \( y(0) = 1 \).
I could not find the exact solution but you may try. Find the numerical solution up to the final time \( T = 4 \) using the same procedure as in the previous problem. Use a value of \( n \) large enough so that the graph of your data looks smooth. Turn in your program with the derivation of the formula you used, and a plot of the solution. You won’t have the error since you don’t know the exact solution.
5. [Required for graduate students, optional for undergraduates] In problem (3) you solved a differential equation by approximating the derivative \( y'(t) \) with a forward difference, which is first-order accurate (i.e. the leading part of the error is \( hK_1 \)). Suppose your advisor wants you to solve the same problem using a second-order method.

For a given \( n \), define \( h = T/n \) and \( t_k = kh \) for \( k = 0, 1, \ldots, n \). Let \( y_k \) represent the numerical solution at \( t_k \). We know that \( y_0 = -1 \) from the initial condition. We can use the second-order approximation

\[
\frac{y_{k+1} - y_{k-1}}{2h} = \sin(t_k), \quad y_0 = -1 \quad \text{for } k = 1, 2, 3, \ldots, n - 1
\]

This gives \( n - 1 \) equations for the \( n \) unknowns \( y_1, y_2, \ldots, y_n \). We cannot use the same equation for \( k = n \) (the last point on the right) since we would need \( y_{n+1} \). Instead, for the point \((t_n, y_n)\) we use a second-order one-sided difference that involves the points \( y_{n-2}, y_{n-1} \) and \( y_n \) (look it up in the book). Once you have \( n \) equations for the \( n \) unknowns, set up a matrix system and solve it in matlab. Compare with the exact solution using \( n = 20 \) and \( n = 40 \). Does the error confirm that the method is second-order?