MATH-121 (DUPRÉ) FALL 2009 TEST 1 ANSWERS

1. QUESTION

From the table given we see that $g(3) = 2$ and $f(2) = 2$, so

$$(f \circ g)(3) = f(g(3)) = f(2) = 2$$

which is answer **A**.

2. QUESTION

We are given $f(x) = x + 3$ and $h(x) = 6x - 2$. We need a function g such that $g \circ f = h$. Notice all the choices are simple linear functions, so the simplest thing to do is just start trying them. The first in the list, choice **A** would be $g(x) = 6x + 20$. If we try this, we find

$$(g \circ f)(x) = 6f(x) + 20 = 6[x + 3] + 20 = 6x + 18 + 20 = 6x + 38$$

and this is NOT $h(x) = 6x - 2$. The next possible choice in the list is $g(x) = 6x - 20$. Using this amounts to changing the sign in front of the 20 in the previous equation:

$$(g \circ f)(x) = 6f(x) - 20 = 6[x + 3] - 20 = 6x + 18 - 20 = 6x - 2,$$

so it now works. The correct answer is therefore **B**.

3. QUESTION

We are given the function $g(u) = \sqrt{u} - \sqrt{8 - u}$ and asked to find its domain. Since no domain has been specified, that means the domain is automatically the largest set of numbers for which the expression makes sense. Thus we only need the entries in the radicals to be non-negative. This means

$$u \geq 0$$

and

$$8 - u \geq 0.$$

These inequalities are equivalent to

$$0 \leq u$$

and

$$u \leq 8,$$

respectively or in short,

$$0 \leq u \leq 8.$$

This means that in interval notation, the domain is

$$[0, 8]$$

which is answer **D**.

4. **QUESTION**

To start we are given

$$f(x) = \frac{5-\sqrt{x}}{25-x}.$$

Its domain is obviously $[0,\infty) \setminus \{25\}$. We are asked to find a function g that agrees with f for $x \neq 25$ and is continuous on $\mathbb{R}$. Actually, none of the answers are correct, but if we ask that g be continuous on $[0,\infty]$, then **C** will work. The main interest here is finding a function which is continuous at $x = 25$ where f is obviously undefined and where f would appear to blow up from a quick glance, if not examined more closely to see that it does not. For x near 25 we can assume that $x > 0$ so $x = |x| = \sqrt{x^2} = (\sqrt{x})^2$, so the denominator factors as

$$25 - x = (5-\sqrt{x})(5+\sqrt{x}).$$

We can therefore cancel the common factors of $5-\sqrt{x}$ that appear in both the numerator and denominator of the expression for $f(x)$ and find that we have

$$f(x) = \frac{1}{5+\sqrt{x}},\ x \neq 25.$$

We can notice that the function

$$g(x) = \frac{1}{5+\sqrt{x}}$$

is defined for all $x \geq 0$, since the denominator can never be zero, so this function is continuous on $[0,\infty)$ and agrees with f for $x \neq 25$. This is answer **C**. It is now easy to give a function which is continuous on all of $\mathbb{R}$ and agrees with f on $[0,\infty) \setminus \{25\}$. Just replace the x in the espression for g by absolute value of x. Thus the correct answer is

$$g(x) = \frac{1}{5+\sqrt{|x|}}.$$

In any case, none of the answers other than **C** are acceptable. The correct answer is that none of the functions are continuous on all of $\mathbb{R}$, but **C** is acceptable as it is easily modified to give the correct answer, and as well satisfies the condition of agreeing with f on its own domain which is $[0,\infty) \setminus \{25\}$.

5. **QUESTION**

A ball is thrown in the air with velocity (initially) 58 ft/sec. Its height $H(t)$ after t seconds is given by $H(t) = 58t - 11t^2$. We are asked to find the velocity when $t = 4$ seconds. The velocity at time t is $H'(t)$ since velocity is the derivative of position. Thus the velocity when $t = 4$ is $H'(4)$. If we use the power and sum rules for differentiation, we see that $H'(t) = 58 - 22t$, and therefore

$$H'(4) = 58 - 22 * 4 = 58 - 88 = -30.$$

As all the answer choices are actually positive numbers, we can surmise the question is actually about the *speed* of the ball which is therefore 30 which is **B**, though technically, the correct answer is none of these. None of the other choices are acceptable.

If we do not use the differentiation rules, we must calculate $H'(4)$ using the limit definition of the derivative which means

$$H'(4) = \lim_{h\to 0} \frac{H(4+h) - H(4)}{h} = \lim_{h\to 0} \frac{58(4+h) - 11(4+h)^2 - [58*4 - 11*4^2]}{h}$$

$$= \lim_{h\to 0} \frac{[58*4 - 11*4^2] + 58h - 11h^2 - 88h - [58*4 - 11*4^2]}{h} = \lim_{h\to 0} \frac{58h - 11h^2 - 88h}{h}$$

$$= \lim_{h\to 0} [-30 - 11h] = -30.$$

6. QUESTION

We are asked to evaluate

$$\lim_{x\to 0^-} f(x)$$

and the function given satisfies

$$f(x) = \sqrt{-x},\ x < 0.$$

This means that we know

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{-x} = \sqrt{-0} = 0,$$

which is answer choice **A**.

7. QUESTION

For the tangent line of to the curve $y = x^3 - 2x$ at the point $(2, 4)$, we know that the tangent line must also pass through the point of tangency which is here $(2, 4)$. None of the answer choices pass through $(2, 4)$, so the answer is none of these, **E**.

8. QUESTION

We need to find the limits

$$\lim_{x\to 9^\pm} F(x),$$

where

$$F(x) = \frac{x^2 - 81}{|x-9|}.$$

We have then on factoring the numerator,

$$\lim_{x\to 9^\pm} F(x) = \lim_{x\to 9^\pm} \frac{(x-9)(x+9)}{|x-9|} = \lim_{x\to 9^\pm} \frac{(x-9)}{|x-9|} \lim_{x\to 9^\pm} (x+9).$$

Notice that

$$\lim_{x\to 9^\pm} \frac{(x-9)}{|x-9|} = \pm 1.$$

On the other hand, $x + 9$ is a continuous function of x so it has a two-sided limit at $x = 9$ and therefore both one-sided limits for $x + 9$ at $x = 9$ exist and are simply given by that two-sided limit:

$$\lim_{x\to 9^\pm} (x+9) = \lim_{x\to 9} = 9 + 9 = 18.$$

Thus,

$$\lim_{x\to 9^\pm} F(x) = (\pm 1)(18) = \pm 18.$$

This is answer choice **B**.

9. **QUESTION**

This question asks us to find where the function $f(x) = |8 - x|$ is not differentiable. If we graph this function, we see that the graph is two lines meeting at right angles forming a V-shape with vertex at the point $(8, 0)$. Thus we see easily that the left-hand derivative is -1 and the right hand derivative is $+1$ at $x = 8$. This means that the two-sided limit for the derivative at $x = 8$ does not exist as the left and right hand limits are not equal. For any $x \neq 8$ the derivative exists, and clearly $f'(x) = +1$, for $x > 8$ and $f'(x) = -1$, for $x < 8$. This is an acceptable answer here even though strictly speaking it does not tell why-it is only a "picture proof".

A more precise proof of the fact that f is differentiable for $x \neq 8$ but not differentiable for $x = 8$, can be given easily by breaking it down into the three cases: (1) for $x > 8$, next (2) for $x < 8$, and finally, (3) for $x = 8$. If $x > 8$, then $f(x) = x - 8$ which is differentiable and therefore f is differentiable with $f'(x) = +1$, for $x > 8$. If $x < 8$, then $x - 8$ is negative and $f(x) = -(x - 8) = 8 - x$, so f is differentiable with $f'(x) = -1$, for $x < 8$. In the final case where $x = 8$, we must compute the derivative limit

$$\lim_{h \to 0} \frac{|8 + h - 8| - |8 - 8|}{h} = \lim_{h \to 0} \frac{|h|}{h},$$

which does not exist as the left and right hand limits are $+1$ and -1, respectively, so the one-sided limits do not agree and therefore the two-sided limit cannot exist.

It is instructive to see why $f(x)$ is differentiable everywhere on $\mathbb{R} \setminus \{8\}$ by computing the limits for the definition of the one-sided derivatives without breaking it up into the various cases. We find

$$D_{\pm}f(x) = \lim_{z \to x^{\pm}} \frac{|z - 8| - |x - 8|}{z - x} = \lim_{z \to x^{\pm}} \frac{(|z - 8| - |x - 8|)(|z - 8| + |z - 8|)}{(z - x)(|z - 8| + |x - 8|)}$$

$$= \lim_{z \to x^{\pm}} \frac{(z - 8)^2 - (x - 8)^2}{(z - x)(|z - 8| + |x - 8|)} = \lim_{z \to x^{\pm}} \frac{(z^2 - 16z + 64) - (x^2 - 16x + 64)}{(z - x)(|z - 8| + |x - 8|)}$$

$$= \lim_{z \to x^{\pm}} \frac{(z^2 - x^2) - 16(z - x)}{(z - x)(|z - 8| + |x - 8|)} = \lim_{z \to x^{\pm}} \frac{(z - x)(z + x) - 16(z - x)}{(z - x)(|z - 8| + |x - 8|)}$$

$$= \lim_{z \to x^{\pm}} \frac{(z + x - 16)}{(|z - 8| + |x - 8|)} = \lim_{z \to x^{\pm}} \frac{(z - 8) + (x - 8)}{(|z - 8| + |x - 8|)}.$$

If $x \neq 8$, these one-sided limits then become

$$D_{\pm}f(x) = \lim_{z \to x^{\pm}} \frac{(z - 8) + (x - 8)}{(|z - 8| + |x - 8|)} = \frac{(x - 8) + (x - 8)}{(|x - 8| + |x - 8|)} = \frac{x - 8}{|x - 8|} = \pm 1,$$

with the plus sign for $x > 8$ and the negative sign for $x < 8$. That is to say, if $x > 8$, then both the left and right hand limits have the same value $+1$, whereas if $x < 8$, then both the left and right hand limits have the same value, namely -1. This means if $x \neq 8$, then the two sided-limit exists and has the value $+1$ if $x > 8$ and has the value -1 if $x < 8$. However, if $x = 8$, the one-sided limits are

$$D_{\pm}f(x) = \lim_{z \to 8^{\pm}} \frac{(z - 8) + (8 - 8)}{(|z - 8| + |8 - 8|)} = \lim_{z \to 8^{\pm}} \frac{(z - 8)}{(|z - 8|)} = \pm 1,$$

and this means that the left and right hand derivatives are unequal so the derivative cannot exist at $x = 8$. That is, we have

$$D_{+}f(8) = +1$$

and

$$D_{-}f(8) = -1,$$

so f' is undefined at $x = 8$.

10. QUESTION

We want to find

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}}{2x-6}.$$

The simple-minded way to think about this is that if x is very near $-\infty$, then x is the negative of a very big number, so x^2-9 is essentially x^2 and $2x-6$ is essentially $2x$. For instance if you compute the value of the expression when x is negative one billion, then clearly there is no use including the 9 or the 6 in the calculation. Thus, the expression very nearly simplifies to

$$\frac{\sqrt{x^2}}{2x} = \frac{|x|}{2x}.$$

We should therefore have

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to-\infty} \frac{|x|}{2x} = -\frac{1}{2}.$$

To be more precise, we can in the original expression divide the numerator and denominator by $|x|$, so we have

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2-9}}{|x|}}{\frac{2x-6}{|x|}}.$$

Then in the numerator, we can replace $|x|$ by its definition $\sqrt{x^2}$ and then combine the radicals in the numerator to a single radical.

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2-9}}{|x|}}{\frac{2x-6}{|x|}} = \lim_{x\to-\infty} \frac{\sqrt{1-9/x^2}}{2(x/|x|-3/|x|)}.$$

Since $x^2 = |x|^2$ and

$$\lim_{x\to-\infty} \frac{x}{|x|} = -1, \ \lim_{x\to-\infty} \frac{1}{|x|} = 0,$$

it follows from the limit rules that

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}}{2x-6} = \frac{\sqrt{1-0}}{2([-1]-0)} = -\frac{1}{2}.$$