MATH-121 (DUPRÉ) FALL 2009 TEST 2 ANSWERS

For problems 1-5, suppose that f, g and h are all functions which are differentiable and that

$$f(5) = 2,\ g(5) = 3,\ h(2) = 4,$$

and

$$f'(5) = 7,\ g'(5) = 11,\ h'(2) = 17.$$

Calculate:

1. ANSWER

$$(f + g)'(5) = f'(5) + g'(5) = 7 + 11 = 18$$

2. ANSWER:

$$(fg)'(5) = f'(5)g(5) + f(5)g'(5) = 7 * 3 + 2 * 11 = 21 + 22 = 43$$

3. ANSWER

$$(f/g)'(5) = \frac{f'(5)g(5) - f(5)g'(5)}{[g(5)]^2} = \frac{21 - 22}{3^2} = \frac{-1}{9}$$

or -.111...

4. ANSWER:

$$(h \circ f)'(5) = h'(f(5))f'(5) = h'(2)f'(5) = 17 * 7 = 119$$

5. The equation of the tangent line to the graph of $h \circ f$ at the point $(5, 4)$.

ANSWER:

$$y = 119(x - 5) + 4$$

For problems 6-9 suppose that $f(x) = e^x g(x)$ where g is differentiable on all of $\mathbb{R}$, and that $g(0) = 1$ and $g'(0) = 3$.

6. Express $f'(x)$ in terms of $e^x, g(x)$, and $g'(x)$.

ANSWER:

$$f'(x) = e^x g(x) + e^x g'(x)$$

7. Calculate $f(0) = e^0 g(0) = 1 * 1 = 1$

8. Calculate $f'(0) = e^0 g(0) + e^0 g'(0) = 1 + 3 = 4$

9. Give the equation of the line tangent to the graph of f at the point $(0, f(0))$, as an equation in x and y.

ANSWER:

$$y = 4x + 1$$

For problems 10-13, DIFFERENTIATE but DO NOT SIMPLIFY the given functions

10.

$$f(x) = \frac{x - \tan x}{\sin x + \sec x}$$

ANSWER:

$$f'(x) = \frac{(1 - \sec^2 x)(\sin x + \sec x) - (x - \tan x)(\cos x + \sec x \tan x)}{(\sin x + \sec x)^2}$$

11.

$$h(x) = (\sin x)^{\cos x}$$

ANSWER:

$$h'(x) = (\sin x)^{\cos x}(\ln[(\sin x)^{\cos x}])'$$
$$= (\sin x)^{\cos x}(\cos x \ln \sin x)'$$
$$h'(x) = (\sin x)^{\cos x}(-\sin x \ln \sin x + \cos x \frac{\cos x}{\sin x})$$

12.

$$g(t) = \sqrt{\frac{t^3 + 7}{1 + t^4}}$$

ANSWER:

$$g'(t) = [\frac{1}{2(\sqrt{\frac{t^3+7}{1+t^4}})}][\frac{3t^2(1 + t^4) - 4t^3(t^3 + 7)}{(1 + t^4)^2}]$$

For problems 13-15 suppose a Lady Bug is crawling on the curve described by the equation

$$x^2 e^{x \sin y} + \tan y = 1,$$

so the Lady Bug's position (x, y) is a function of time t, and as a result x and y are functions of time, t. We call $\dot{x}(t)$ the Lady Bug's horizontal velocity at time t, and we call $\dot{y}(t)$ the Lady Bug's vertical velocity at time t.

13. If the equation relating $x, y, \dot{x}$, and $\dot{y}$ is expressed as $F(x, y)\dot{x} + G(x, y)\dot{y} = 0$, where $F(x, y)$ and $G(x, y)$ are functions of the variables x and y, then express the functions F and G in terms of x, y, using power, trig, and exponential functions.

We begin by differentiating the left side of the equation with respect to time t. Using overdots to denote time differentiation,

$$(x^2e^{x\sin y}+\tan y)^\bullet = 2x\dot{x}e^{x\sin y}+x^2e^{x\sin y}(x\sin y)^\bullet+(\sec^2 y)\dot{y}$$

$$= 2x\dot{x}e^{x\sin y}+x^2e^{x\sin y}(\dot{x}\sin y+x[\cos y]\dot{y})+(\sec^2 y)\dot{y}$$

$$= [e^{x\sin y}(2x+x^2\sin y)]\dot{x}+[e^{x\sin y}x^3\cos y+\sec^2 y]\dot{y}.$$

Thus,

$$F(x,y)=e^{x\sin y}(2x+x^2\sin y)$$

and

$$G(x,y)=e^{x\sin y}x^3\cos y+\sec^2 y$$

You might also notice here that if you differentiate $f(x,y)$ with respect to x treating y as a constant, then you rapidly find $F(x,y)$ whereas if you differentiate $f(x,y)$ with respect to y treating x as a constant, then you rapidly find $G(x,y)$. More about this in general follows below.

We have now found our equation relating $\dot{x}$ and $\dot{y}$ is $F\dot{x}+G\dot{y}$ where F and G are given above. At the point $(1,0)$ we have $x=1$ and $y=0$, so $F=F(1,0)=2$ and $G=G(1,0)=2$, because $\sin 0=0$ whereas $\cos 0=\sec 0=1$. We therefore have $2\dot{x}+2\dot{y}=0$ and therefore simply $\dot{y}=-\dot{x}$ at time t where the Lady Bug is at the point $(1,0)$.

14. If the Lady Bug arrives at the point $(1,0)$ at time $t=3$ with horizontal velocity $\dot{x}(3)=4$, then what is the Lady Bug's vertical velocity, $\dot{y}(3)$, at the instant of arrival at $(1,0)$?

ANSWER: $\dot{y}(3)=-4$

15. Give dy/dx in terms of x and y for the curve along which the Lady Bug is traveling.

We have $F\dot{x}+G\dot{y}=0$ and therefore

$$\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}=-\frac{F(x,y)}{G(x,y)}=-\frac{e^{x\sin y}(2x+x^2\sin y)}{e^{x\sin y}x^3\cos y+\sec^2 y}.$$

We can notice here that whenever we have an equation $f(x,y)=c$ where c is a constant and f is a function of the two variables x and y, that we can think of x and y as depending on time in any way we want. Of course, to remain on the curve specified by the equation, we must have relations between the way x and y depend on t, but as far as the differentiability and differentiation of $f(x,y)$ with respect to t are concerned, we can allow any (differentiable) dependence of x and y on time. We notice that no matter what the dependence of these variables x and y is on t, we will always have as a result of our differentiation rules that we can arrange the time derivative of $f(x,y)$ in the form

$$[f(x,y)]^\bullet = F(x,y)\dot{x}+G(x,y)\dot{y}$$

where F and G are some functions of the variables x and y. Since this would be the case no matter how x and y depend on t, we can take the case that $x=t$ and y is constant independent of t. The result of such a differentiation is that we are differentiating with respect to $x=t$ and y is constant. But then we have $\dot{x}=1$ and $\dot{y}=0$. If we put these into the equation above for $\dot{f}$, we see it immediately reduces to $\dot{f}=F$. Likewise, if we take the case where the dependence of x and y on t is x is constant independent of t and $y=t$, then $\dot{x}=0$ and $\dot{y}=1$, and the equation for $\dot{f}$ reduces to $\dot{f} = G$. Thus, we can find F by differentiating f with respect to x and treating y as a constant and also find G by differentiating f with respect to y treating x as a constant. Once we have the functions F and G, the equation

$$\dot{f} = F\dot{x} + G\dot{y}$$

is then valid no matter what the dependence of x and y on t, and in particular, must be true for the dependence of x and y on t for the motion along the curve $f(x, y) = c$, whatever it may be.