MATH-121 (DUPRÉ) FALL 2009 TEST 3 ANSWERS

For problems 1-3 give all the critical points of the specified function f.

1. $f(x) = 2x^3 - 3x^2 - 36x + 2$

ANSWER: Calculating f' we see

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x+2)(x-3),$$

so the only solutions of $f'(x) = 0$ are $x = -2$ and $x = 3$. Also the domain of f' is the set of all real numbers, so the only critical points are $x = -2$ and $x = 3$.

2. $f(x) = ax^2 + bx + c$, where $a, b,$ and c are constants with $a \neq 0$. Express your answer in terms of $a, b,$ and c.

ANSWER: Calculating f' we have

$$f'(x) = 2ax + b = 2a(x + \frac{b}{2a}),$$

so the only solution of $f'(x) = 0$ is $x = -b/[2a]$. Also the domain of f' is the set of all real numbers, so the only critical point is $x = -b/[2a]$.

3. f is the antiderivative of the function $g(x) = x(x+2)(x-1)(x-4)$ such that $f(0) = 5$.

ANSWER: As f is an antiderivative of g, we know, by definition, this means $f' = g$, so

$$f'(x) = g(x) = x(x+2)(x-1)(x-4),$$

and therefore the only solutions of $f'(x) = 0$ are $x = -2, 0, 1, 3$. Since the domain of $g = f'$ is all real numbers, the only critical points of f are the four points $x = -2, 0, 1, 3$.

4. The function g is $g(x) = ax^3 + bx^2 + cx + d$, where $a, b, c,$ and d are constants with $a \neq 0$. Suppose that g has an inflection point at $x = k$. Express k in terms of $a, b, c,$ and d.

ANSWER: Since g' and g'' are defined for all real numbers, the inflection points are simply the solutions of $g''(x) = 0$. We calculate

$$g'(x) = 3ax^2 + 2bx + c$$

and

$$g''(x) = 6ax + 2b = 6a(x + \frac{2b}{6a}) = 6a(x + \frac{b}{3a}),$$

so the only solution is $x = -b/[3a]$, and therefore

$$k = \frac{-b}{3a}.$$

5. Find the critical points of the function $f(x) = \sqrt{|x|}$.

ANSWER: Define the function s by $s(x) = |x|$. Then we know

$$s'(x) = \frac{x}{|x|} = \frac{|x|}{x}, \ x \neq 0,$$

and s' is undefined at $x = 0$. We have $f(x) = \sqrt{s(x)}$, so by the Chain Rule,

$$f'(x) = \frac{1}{2s(x)} s'(x) = \frac{1}{2\sqrt{|x|}} \frac{|x|}{x} = \frac{\sqrt{|x|}}{2x}, \ x \neq 0.$$

This formula for f' shows that if $x \neq 0$, then $f'(x) \neq 0$. On the other hand, the formula also shows that $x = 0$ is not in the domain of f' but it is certainly in the domain of f, so $x = 0$ is the only critical point of f.

Alternately, we can restrict attention to the sets $x \geq 0$ and $x \leq 0$ and calculate the derivative on each separately. We find

$$f(x) = \sqrt{x},\ x \geq 0,$$

so

$$f'(x) = \frac{1}{2\sqrt{x}},\ x > 0,$$

and the right hand derivative of f at $x = 0$ is undefined. For $x \leq 0$ we have

$$f(x) = \sqrt{-x},\ x \leq 0,$$

so

$$f'(x) = \frac{1}{2\sqrt{-x}}(-1),\ x < 0,$$

and the left hand derivative of f at $x = 0$ is undefined. Thus, if $x \neq 0$, we have $f'(x) \neq 0$, whereas $x = 0$ is in the domain of f but not in the domain of f', so $x = 0$ is the only critical point of f.

6. Find an antiderivative F for the function $f(x) = 12x^5 + \dfrac{3}{x}$ with $F(1) = 4$.

ANSWER: We can calculate

$$F(x) = \int f(x)dx = \int (12x^5 + \frac{3}{x})dx = 12\int x^5 dx + 3\int \frac{1}{x}dx = 12\frac{x^6}{6} + 3\ln|x| + C,$$

where C is a constant. Thus, we can simplify and find

$$F(x) = 2x^6 + 3\ln|x| + C,$$

and we can find the value of the constant C using the condition that $F(1) = 4$. We have

$$4 = F(1) = 2(1^6) + 3\ln(1) + C = 2(1) + 3(0) + C,$$

so $4 = 2 + C$ and therefore $C = 2$. Using this value for C we finally have

$$F(x) = 2x^6 + 3\ln|x| + 2.$$

7. Calculate $\displaystyle\int_0^1 x^{22.7}dx =$

ANSWER: For any value of $p \neq -1$, the power rule for antidifferentiation gives

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C,$$

and therefore

$$\int_0^1 x^p = \frac{x^{p+1}}{p+1}|_0^1 = \frac{1}{p+1}.$$

In particular, this means

$$\int_0^1 x^{22.7}dx = \frac{1}{23.7} = \frac{10}{237}.$$

8. Find the indefinite integral $\int \frac{\cos x}{\sin^5 x} dx$.

ANSWER: The integrand is

$$\frac{\cos x}{\sin^5 x} = \frac{\cos x}{(\sin x)^5},$$

so we see that its denominator is a function of a function. The first choice here should be to try to substitute $u = \sin x$. If we do this, then $du = \cos x dx$, so we have

$$\int \frac{\cos x}{\sin^5 x} dx = \int \frac{du}{u^5}|_{u=\sin x} = \int u^{-5} du|_{u=\sin x} = \frac{u^{-4}}{-4}|_{u=\sin x} + C = C - \frac{1}{4\sin^4 x}.$$

9. Find the area above the $x-$axis and underneath the curve $y = 4 - x^2$ between the limits $x = 1$ and $x = 2$.

ANSWER: If we denote the area asked for by A, then

$$A = \int_1^2 (4 - x^2) dx = [4x - \frac{x^3}{3}]_1^2 = (8 - \frac{8}{3}) - (4 - \frac{1}{3}) = 4 - \frac{7}{3} = \frac{5}{3}.$$

10. Calculate $\int_0^3 6|(x-1)(x-2)| dx =$

ANSWER: Whenever we must calculate $\int_a^b |f(x)| dx$, remember we must solve the equation $f(x) = 0$ and find ALL solutions in the interval $[a, b]$. Since in our problem here we have $f(x) = 6(x-1)(x-2)$, we see that the only solutions of $f(x) = 0$ in the interval $[0, 3]$ are $x = 1$ and $x = 2$. Therefore,

$$\int_0^3 6|(x-1)(x-2)| dx = |\int_0^1 f| + |\int_1^2 f| + |\int_2^3 f|,$$

that is to say, we now have the absolute value symbols outside the integral for each term on the right, so we can now easily find the antiderivative of f to perform the definite integrals. We have

$$f(x) = 6(x-1)(x-2) = 6(x^2 - 3x + 2),$$

so denoting an antiderivative of f by F, we can take F to be

$$F(x) = 6(\frac{x^3}{3} - 3\frac{x^2}{2} + 2x) = 2x^3 - 9x^2 + 12x.$$

To calculate the definite integrals on the right hand side of the integral equation, we simply need the values of F at $x = 0, 1, 2, 3$. These values are

$$F(0) = 0,$$
$$F(1) = 2 - 9 + 12 = 5,$$
$$F(2) = 2(2^3) - 9(2^2) + 12(2) = 16 - 36 + 24 = 4,$$
$$F(3) = 2(3^3) - 9(3^2) + 12(3) = 2(9)(3) - 9(9) + 36 = 3[18 - 27 + 12] = 3(3) = 9.$$

In short, we have $F(0) = 0,\ F(1) = 5,\ F(2) = 4,\ F(3) = 9$, so
$|\int_0^1 f| = |5-0| = 5,\ |\int_1^2 f| = |4-5| = 1,\ |\int_2^3 f| = |9-4| = 5$, and this means finally,

$$\int_0^3 6|(x-1)(x-2)|dx = 5+1+5 = 11.$$

Notice here that if you simply ignore the absolute value signs in the integrand and calculate instead $\int_0^3 f$, then you get

$$\int_0^3 6(x-1)(x-2)dx = \int_0^3 f = \int_0^1 f + \int_1^2 f + \int_2^3 f = 5 + (-1) + 5 = 9 = F(x)|_0^3,$$

and this equation shows precisely where ignoring the absolute value signs goes wrong, as the middle integral is negative without the absolute value. To repeat here, the correct answer is then 11 not 9,

$$\int_0^3 6|(x-1)(x-2)|dx = 11.$$

11. A rectangle in the first quadrant has two of its edges on the x and y axes and one vertex on the curve $y = 12 - x^2$. Find the maximum area possible for the rectangle.

ANSWER: We imagine here that x and y can vary with time along the curve $y = 12 - x^2$, so the area of the rectangle $A = xy$ is then also depending on time t. We denote time derivatives by overdots, so

$$\dot{A} = \dot{x}y + x\dot{y}$$

by the Product Rule and

$$\dot{y} = -2x\dot{x},$$

by the Chain Rule. Substituting the expression for $\dot{y}$ in terms of x and $\dot{x}$ given by the second equation into the first for $\dot{A}$, we find

$$\dot{A} = y\dot{x} + x(-2x\dot{x}) = [y - x^2]\dot{x}.$$

We can assume that we allow x to vary with time so as to make $\dot{x} \neq 0$ at the optimal value of x, and this means the equation $\dot{A} = 0$ is equivalent to

$$y - 2x^2 = 0,$$

or $y = 2x^2$. But on the curve $y = 12 = x^2$, we then have

$$2x^2 = y = 12 - x^2$$

so

$$3x^2 = 12$$

and therefore $x^2 = 4$ so $x = 2$. Since $y = 12 - x^2$, this means for the optimal rectangle, $y = 12 - 2^2 = 12 - 4 = 8$, or $y = 8$. Therefore, the optimal rectangle is the rectangle with $x = 2$ and $y = 8$ and therefore with area $A = 16$.

Alternately, here you can use the equation $y = 12 - x^2$ right at the start to substitute the value of y in terms of x into the expression for area A. The result is

$$A = xy = x(12 - x^2) = 12x - x^3 = A(x),$$

so now A is a function of x alone. To find the optimal rectangle, we just differentiate A with respect to x and set the derivative equal to zero. We find

$$A'(x) = 12 - 3x^2$$

so the equation $A'(x) = 0$ gives $3x^2 = 12$ and therefore again $x = 2$, and $A(2) = 16$, for the maximum area.

Notice that technically, we have only found the local extreme value of the area, but obviously, $A \geq 0$ for the rectangle described in the problem, and $A = 0$ if either $x = 0$ or $y = 0$. But our equations show there is only one optimal value between $x = 0$ and $x = \sqrt{(12)}$ where $y = 0$, namely the value when $x = 2$ which is $A = 16$, so it must be the absolute maximum value.

12. Calculate $\lim_{x\to 0} \dfrac{\sin(6x^2)}{3x^2} =$

ANSWER: We notice that this limit has the form 0/0 so by L'Hospital's Rule,

$$\lim_{x\to 0} \frac{\sin(6x^2)}{3x^2} = \lim_{x\to 0} \frac{12x\cos(6x^2)}{6x} = \lim_{x\to 0} 2\cos(6x^2) = 2\cos(0) = 2.$$

13. Calculate $\lim_{x\to 0} \dfrac{3x^2+5}{\cos(6x^2)} =$

ANSWER: This limit is NOT of any form for which L'Hospital's Rule applies. In fact, the function

$$f(x) = \frac{3x^2+5}{\cos(6x^2)}$$

is perfectly continuous at $x = 0$, so to compute the limit, simply substitute $x = 0$ in the function. The result is

$$\lim_{x\to 0} \frac{3x^2+5}{\cos(6x^2)} = \frac{5}{\cos(0)} = \frac{5}{1} = 5.$$

14. Calculate $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x} =$

ANSWER: This limit is obviously of the form ∞/∞, so L'Hospital's Rule applies and

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{(2\ln x)(1/x)}{1} = \lim_{x\to\infty} \frac{2\ln x}{x},$$

which is again of the form ∞/∞. Therefore, we apply L'Hospital's Rule again. The result is

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} = \lim_{x\to\infty} \frac{2(1/x)}{1} = \lim_{x\to\infty} \frac{2}{x} = 0.$$