

THE COX THEOREM UNKNOWNNS AND PLAUSIBLE VALUE

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ABSTRACT. We give a proof of Cox's Theorem on the product rule and sum rule for conditional plausibility without assuming continuity or differentiability of plausibility. Instead, we extend the notion of plausibility to apply to unknowns giving them plausible values.

1. INTRODUCTION

Since the work of Laplace [17] in the late 18th century, there have been many attempts by mathematicians to axiomatize probability theory. The most important example in the 20th century was that of A.N. Kolmogorov [14], who gave a very simple measure-theoretic set of axioms that modeled the view of probability introduced into quantum mechanics by Max Born in 1927. Remarkably, most physicists, in their non-quantum applications of probability, have not followed Born or Kolmogorov but R. T. Cox, who in turn based his approach on Laplace's original idea that probability theory is a precise mathematical formulation of plausible reasoning. These physicists argue that, while the Kolmogorov axioms are elegant and consistent, they are much too limited in scope. In particular, the Kolmogorov axioms in their original form do not refer to conditional probabilities, whereas most physics applications of probability theory require conditional probabilities. Even though unknown by most mathematicians who work in probability theory, the Laplace-Cox approach to probability theory was actually accepted by many distinguished mathematicians prior to Kolmogorov, for examples, Augustus de Morgan [19], Emile Borel [1], Henri Poincaré [21], and G. Pólya [22]. For a discussion of applications of Laplacian probability in the foundations and interpretation of quantum mechanics see Tipler [26].

Cox's probability theory is not defined by precise axioms, but by three "desiderata": (I) representations of plausibility are to be given by real numbers; (II) plausibilities are to be in qualitative agreement with common sense; and (III) the plausibilities are to be "consistent", in the sense that anyone with the same information would assign the same real numbers to the plausibilities. Cox ([2],[3], pg. 16) purported to show that from these requirements, the plausibilities satisfied, first the **PRODUCT RULE**:

$$PL(A\&B|C) = PL(A|B\&C)PL(B|C),$$

and the **SUM RULE**

$$PL(A|B) + PL(\bar{A}|B) = 1$$

The claim that these two rules follow from the desiderata has come to be known as **COX'S THEOREM**. The symbol $PL(A|B)$ means a conditional plausibility, namely "the plausibility of A given that we know B ." The symbol "&" represents the logical "both," whereas the bar on top represents logical negation.

We shall give in this paper a rigorous mathematical proof for Cox's Theorem on the product rule for conditional plausibility of propositions as used in plausible reasoning, a proof that follows from precise axioms. We shall see that our axioms are mathematically simpler and more intuitive than Cox's desiderata. In particular, we shall not need to make any continuity or differentiability assumptions. It is very important to avoid assuming continuity if the symbols A and B refer to propositions — as they do in Cox's paper and book and as they do in Jaynes'

important book *Probability Theory* — because propositions are necessarily constructed from a finite number of symbols, and hence properly belong to the integers and not to the continuum (as represented, for example, in the Gödel numbering scheme in the proof of the Gödel theorems). We will not follow Kolmogorov and list a short and ideal set of axioms from which all of probability theory can be derived, but instead give a list of axioms and possible alternatives for several. All of our alternatives are much less technical and more intuitive than those of such authors as Halpern's for example.

In order to provide this very simple set of axioms for proving the Cox Theorem and deriving the rules of probability so as to make them apparent to even a reader without expert mathematical training, we are led to expand the objective of plausibility theory to more generally deal with objects we call unknowns which have plausible values. The aims of the theory of plausible reasoning are two-fold. First the aim is to derive the rules by which logic and common sense constrain our inductive reasoning in the face of limited information, and second to derive the rules of probability from simple assumptions, so as to make them apply to propositions in general. A major motivation in our paper is to make probability applicable in scientific settings where the frequency theory of probability is of little or no value, and to justify a Laplacian or Bayesian approach to probability ([13],[25],[15],[23]). The assumptions need to be well motivated and very simple, and the proof of the basic rules of probability from these assumptions should hopefully be trivial. As the counterexample of Halpern [8] shows, the original assumptions of Cox are inadequate, and the technical assumptions of Paris [20] are undesirable and still require an unjustifiable continuity assumption. More recently, the work of Hardy ([10], Theorem 8.1) shows very generally that with sufficient hypotheses the theorem is true, but the hypotheses on the range of values could be problematic to verify in practice, even though the development is important and nontrivial.

In brief, in the standard approach, one assumes a Boolean algebra E of propositions together with a real valued function $PL(A|B)$ defined for $(A, B) \in E \times E_0$, where $E_0 = E \setminus \{0\}$ and which we think of $PL(A|B)$ as assigning a numerical level of **PLAUSIBILITY** to proposition A given that we accept proposition B as true. We further assume that $PL(A|B)$ is a monotonic function of the plausibility of A when B is assumed true. Consequently, we are allowed to modify PL by composing with a monotonic function if necessary to produce a useful rule. The first result of the standard approach is that allowing such modifications we can produce the product rule and the sum rule. In the original proof of the product rule, which is essentially the rule for conditional probability, R.T. Cox (see [3], page 12) assumed merely that the plausibility $PL(A&B|C)$ was a numerical function of the plausibilities $PL(A|B&C)$ and $PL(B|C)$ through some real valued function F of two variables. Motivating this assumption requires examination of a host of special cases [27] for the different possibilities of what $PL(A&B|C)$ could depend on among the four numbers

$$PL(A|B&C), PL(A|C), PL(B|A&C), PL(B|C),$$

an examination rendered unnecessary in the approach we will introduce here. Then evaluating $PL(A&B&C|D)$ in the two possible ways available and applying associativity of the conjunction of propositions almost leads to the conclusion that the function F is an associative multiplication on the set of real numbers forming the range of PL . This last step taken by Cox was a logical mistake as the counterexample of Halpern [8] shows, this conclusion is not justified as E may be finite, and even if it were true, there would in general be no useful information coming from this fact. But Cox assumed that the function F should be of a universal character and therefore must be defined on the whole plane. Cox thus assumed F is an associative multiplication on an interval of real numbers. Assuming the function to be differentiable leads to the assumed multiplication being in fact ordinary multiplication. However, the assumption that the function is differentiable was never justified by either Cox or Jaynes, except by hand waving. Moreover, the domain of the function may in reality only be a finite set of real numbers, and so extreme effort has gone into trying to add on very technical assumptions [20], [8], which in effect produce

sufficient density of the domain to claim that continuity gives associativity which together with strict monotonicity (a requirement from "agreement with common sense") suffices to show that the associative multiplication is just ordinary multiplication.

However, it has been well known for many many years by experts in the theory of topological semigroups, what possible continuous multiplications are available on an interval of real numbers. Numerous textbooks in topological semigroup theory address this very issue. As shown for example in the seminal work by K.H. Hofmann and P.S. Mostert [12], the possibilities are infinite. However, if we assume the strict monotonicity which seems consistent with common sense and which rules out idempotents other than a zero and a unit to form the boundary of the interval, then the only continuous associative multiplication is isomorphic to the unit interval under ordinary multiplication. Thus a suitable function of PL would then satisfy the multiplication rule, a result which properly belongs to the theory of topological semigroups. Several authors have dealt with counterexamples [5], [8], [16] and proofs [20], in effect reproving results of topological semigroup theory, and the complete proof of Cox's Theorem, even with the assumption of continuity, is not simple. In the case of Hardy [10], we have a fairly complete theory of scales which in effect provide alternate density type assumptions on the set of values of the plausibility function ([10], Theorem 8.1). The scales are themselves lattices of special type which under the proper technical assumptions are shown isomorphic to the unit interval. This approach is very general and in spirit similar to the (noncommutative operator algebra) case treated by Loomis [18]. Moreover, these technical assumptions such as continuity or divisibility are just as problematic as the assumption of differentiability. That is, they are certainly reasonable, and not as strong as assuming differentiability, but in the end, they are still strong and highly technical, non-intuitive assumptions. Once the Cox Theorem is proved, the modified function PL , under another common sense assumption, namely that $PL(notA|B)$ depends only on $PL(A|B)$, can be shown to have (at least a power, depending on which axioms are used) which obeys the laws of probability. Several authors have dealt with the problem of associativity of the universal function, required for the Cox Theorem, as it seems to be essential to the argument given by Cox, and it is essentially a result of topological semigroup theory which is being applied by all these authors. However, we will see that in our approach, questions of continuity or associativity become completely irrelevant to the argument.

Our simpler approach takes a closer look at what scientists are really trying to do. The main aim of scientists, engineers and technical workers is arriving at values for numerical quantities on the basis of limited information. Thus, instead of restricting attention to a set of propositions, we are led instead to consider a set of more general objects we shall call **UNKNOWNNS**. We purposefully do not use the term "random variable" here, as it is a much too restrictive a notion, and carries with it all the baggage of the Kolmogorov approach to probability theory, but a random variable is an example of an unknown. In case of propositions, since all members of a Boolean algebra are idempotent, and as the only idempotent numbers are 0 and 1, we are naturally lead to create or define an unknown number, I_A , for each proposition A called its **INDICATOR**. Our object now is to assign a **PLAUSIBLE VALUE** denoted $PV(X|A)$ to the unknown X given the information in proposition A . As for plausibility of statements, we then simply define $PL(A|B) = PV(I_A|B)$. The result is we find a very simple and natural theory of plausible value for unknownns which contains the theory of plausibility of propositions and which requires no assumptions at all in the form of differentiability or even of continuity for its rules. The rules are simply dictated by simple common sense consistency with logic. The main idea turns out to be exceedingly simple and really only depends on some simple properties of retraction mappings on sets. What comes out of these considerations is that the rules are really uniquely determined, in a very strong sense, merely by the assumption that some form of rule exists. In short, existence implies a strong form of uniqueness. We begin with simple considerations of retraction mappings on sets, and then when we get to the setting of unknownns, we see right away that the PV must be a retraction of the unknownns onto the knownns. Thus, the assumption of the existence of rules of dependency of certain

general forms can be completely determined by what happens to the known quantities under the general forms of the rules. In particular, if we examine what this approach does for the plausibility theory, we note that a natural logical axiom of rescaling of plausible value under changes of units causes the universal function of the Cox theorem proof to be homogeneous in its first variable. This axiom for plausibility means that plausibility should really be a geometric quantity which is independent of the choice of maximum and minimum. That is, we should think of the plausibility of a statement as being specified by a point on a line segment where one endpoint is the plausibility of a known true statement and the other endpoint is the plausibility of a known false statement. That geometric picture is independent of the numerical scale chosen for the segment, and a realistic plausibility theory should contain that property. That is, if someone asks you what is the plausibility of statement A given statement B is true on a scale of a to b , you should be able to express the plausibility on that scale demanded no matter what scale you had originally chosen to express plausibility. What this means is that if we define $O(A|B) = PL(A|B)/PL(notA|B)$, usually called the odds of A given B , then $O(A|B)$ is completely scale invariant. Homogeneity of the universal function of the Cox theorem gets around the counterexample of Halpern [8]. In fact by Halpern's theorem 3.1 and lemma following, if F satisfying the conclusion of his theorem is homogeneous in the first variable, then we find immediately that $F(x, y) = xy$ as an immediate consequence of his theorem 3.1, so F is associative, a contradiction of his following lemma. His construction technique is to take a finite set of 12 members and by using two slightly different probability distributions, join them in an unnatural way to produce a plausibility theory which satisfies the assumptions of Cox but for which the universal function F cannot possibly be associative because of the way the two probability distributions are joined to produce the plausibilities. Of course, we see immediately now, that Halpern's counterexample violates the natural rescalability that plausibility should have, that is, his function F cannot be homogeneous in its first variable, so his counterexample fails to be a counterexample in any system of plausibility theory in which plausibilities have a natural scale invariant meaning.

2. SIMPLE RETRACTION PRINCIPLES

One of the first things a mathematics student learns is that if f and g are functions on the set T , if g has range S so that $g(T) = S$, then there is at most one function h with domain S satisfying $f = hg$. In short, for such h to exist, clearly $f(t)$ as a function of $t \in T$ must only depend on the value $g(t)$, or in other words, if $g(t_1) = g(t_2)$, then $f(t_1) = f(t_2)$. If we assume this condition is satisfied, then using the axiom of choice if necessary, we can form a **SECTION** of g , namely a function s from S to T with the property that $gs = id_S$, the identity function on S . We get h on setting $h = fs$. For then, $hg = fsg$, but $gsg = g$ implies $fsg = f$, by the assumed condition. In a sense here, we can say existence implies uniqueness, but the function h we find does not have a simple dependence on f for its construction. We may have to use the axiom of choice. We will see that the dependence of h on g is quite explicit if g is a retraction onto a subset of T .

To begin, recall that if T is any set, $R \subset T$ is any subset of T , then, a **RETRACTION** P of T onto R is a self mapping of T such that its image is R and $P(x) = x$ for each $x \in R$. We shall also find it useful to recall the idea of a **RESTRICTION** of a function: if f is a function defined on T , then we denote by $f|R$ its restriction to the subset R , that is the same rule, but with domain restricted to be R .

Proposition 2.1. *Suppose that P is a retraction of the set T onto the subset R and that f is a function from T to set S . If $f(t)$ for $t \in T$ only depends on the value $P(t)$, then there is a unique function h defined on R with $f = hP$, and in fact $h = f|R$, the restriction of f to R .*

Proof. The hypothesis that $f(t)$ only depends on $P(t)$ guarantees the existence of h . But now, for $r \in R$, we have $P(r) = r$ as P is a retraction onto R , and hence $f(r) = h(P(r)) = h(r)$, so $h = f|R$. \square

Corollary 2.1. *Suppose that P_k is a retraction of the set T_k onto the subset R_k , for $k = 1, 2, 3$. Suppose m is a mapping from $T_1 \times T_2$ into T_3 with $m(R_1 \times R_2) \subset R_3$, and denote this mapping by juxtaposition, $m(x, y) = xy$. Then:*

(1) *if f is a function from T_1 to T_2 with $f(R_1) \subset R_2$, and if $P_2(f(t))$ depends only on $P_1(t)$, then*

$$(2.1) \quad P_2(f(t)) = f(P_1(t)), \quad t \in T_1.$$

(2) *if $P_3(t_1 t_2)$ depends only on $(P_1(t_1), P_2(t_2))$, then*

$$(2.2) \quad P_3(t_1 t_2) = P_1(t_1)P_2(t_2), \quad (t_1, t_2) \in T_1 \times T_2;$$

(3) *if in (2) we have a fixed $e \in T_2$ and if we instead assume that $P_3(t_1 e)$ depends only on $P_1(t_1)$, then*

$$(2.3) \quad P_3(t_1 e) = P_3([P_1(t_1)]e), \quad t_1 \in T_1;$$

(4) *if for (3) in addition we assume e has the property that $P_3(re) = rP_2(e)$, for all $r \in R_1$, then*

$$(2.4) \quad P_3(t_1 e) = P_1(t_1)P_2(e), \quad t_1 \in T_1.$$

Proof. The hypothesis in (1) guarantees a function h defined on R_2 with the property that $P_2 f = hP_1$. But now the proposition tells us that $h = P_2 f|_{R_1}$, but $P_2 f|_{R_1} = f|_{R_1}$, because $f(R_1) \subset R_2$ and P_2 is a retraction onto R_2 . The hypothesis in (2) guarantees that $P_1 \times P_2$ is a retraction of $T_1 \times T_2$ onto $R_1 \times R_2$, and hence using (1) with $f = m$ completes the proof for (2). In case of (3), with $e \in T_2$ fixed, we have a unique function h_e from R_1 to R_3 such that $P_3(t_1 e) = h_e(P_1(t_1))$, for all $t_1 \in T_1$. But then, taking $r \in R_1$, we have $P_1(r) = r$, so

$$h_e(r) = h_e(P_1(r)) = P_3(re),$$

and [2.3] follows immediately. Now, (4) is clear from (3). □

In particular, if we take $P_1 = P_2 = P$ in (1) of the corollary, then we see that $P(f(t))$ depends only on $P(t)$ exactly when $Pf = fP$, a **GENERAL COMMUTATION RULE**. In case of (2), we have a **GENERAL COMBINATION RULE**: if $P_3(xy)$ depends only on $(P_1(x), P_2(y))$, then $P_3(xy) = P_1(x)P_2(y)$. On the other hand, if we take the case where $T_1 = R_1$, so P_1 is simply the identity on R_1 , then when $P_3(ry)$ depends only on $(r, P_2(y))$ for $r \in R_1$ we conclude from (2) that $P_3(ry) = rP_2(y)$, a form of **GENERAL HOMOGENEITY**. We can also conclude this for fixed r in R_1 using (1). That is, we take f above to be left multiplication by $r \in R_1$. We can note that (4) above is a very general form of the product rule part of Cox's Theorem. In particular, we note that the question of any form of associativity never enters the proof of (4).

3. UNKNOWNNS AND PROPOSITIONS

Scientists, engineers, and technical workers deal with a world of numbers, and other mathematical entities many of which are not completely known. In many situations, when the description of a particular quantity's numerical value tells us only that a well defined value exists without telling us what it actually is, we must proceed with a most plausible value based on the information at hand which may be incomplete, and which may not be certain. The information generally appears as a proposition which in fact is either true or false, and once accepted is assumed true for purpose of evaluating the unknown quantity as well as we can.

Such quantities are actually more than simple real numbers, as their descriptive information is part of their structure and does not generally give us enough information to determine a certain value. Thus, we can consider them to be objects in some set containing the set of all real numbers and that there is some real valued function on that set which gives each object a value and that this function is unknown to us. We wish to analyze how the requirement of logical consistency constrains the procedure for arriving at plausible values for these objects or unknown quantities when limited information is available. Even if we are just guessing, their should be certain simple logical constraints. As Cox [3] has shown, if we try to apply plausibility with no information, we arrive at absurd results, so our prior information must give us some information about an unknown of interest. More generally, scientists and engineers often deal with mathematical structures beyond the real number system and the same considerations apply. When a physicist speaks of the state Ψ of a classical bounded quantum mechanical system, he generally means that $\Psi \in H$, where H is some Hilbert space, but before he applies the rules of quantum mechanics, he really does not know what Ψ is. In fact, he may not even know what H is. In fact, he may not know enough about the actual physical system for the rules of quantum mechanics to determine what Ψ is. He assumes by the axioms of quantum mechanics that the physical system under consideration determines a unique state, but the information and measurements he actually has for the system may not be enough to actually determine Ψ . For instance, Ψ could be the state of a black cat in a closed box which we cannot see inside, but which we can hear meowing. We could therefore properly think of Ψ as a symbol for an unknown unit vector in H , and we could try, based on C , the proposition stating the measurements we have made and our knowledge of quantum mechanics, to arrive at a plausible value $PV(\Psi|C) \in H$. The same type of consideration applies to any unknown member of any set based mathematical structure. Information can appear in the form of differential equations which must be satisfied as well as experience we have in dealing with similar problems in our past—everything we know can be brought to bear on the choice of a plausible value. When the mathematical structure has rules of combination such as vector addition, semigroup multiplication, actions of one system on another, and so forth, clearly these same operations should apply to the unknowns. Thus, if X is an unknown number and Ψ is an unknown vector in H , then $X\Psi$ is another unknown vector in H . If we are interested in the unknown Ψ in H and the unknown Φ in H , then we possibly we could end up needing to consider $\Psi + \Phi$. Certainly if we have information about each of the summands, then we know something about the sum. Thus it is reasonable to assume that whatever unknowns we are interested in dealing with algebraically form the same kind of system as the system they "live in". For instance, we could think of the Hilbert space H as being an unknown member of a small category of Hilbert spaces if it is also unknown.

To begin, let us be precise about our set up and then consider examples of what we mean by an **UNKNOWN**. Suppose that S is any set. Suppose that B is a proposition which describes a member X of S sufficiently well so that B implies such a member exists even though B might not state which member of S it is, then X is an unknown member of S . In particular, if $s \in S$, then we regard s as known, that is, a known unknown. Thus, if we are interested in a set T of unknown members of S , then we usually assume that $S \subset T$. That is to say, we should think of the unknowns in S as having additional structure by virtue of their descriptions, and we regard the known members of S as contained in the unknowns. To proceed formally, then we will simply assume that $S \subset T$ are sets and we are regarding T as the set of unknowns of S in which we are interested. Of course, as each $X \in T$ is an unknown member of S , it must have a value $AV(X)$, called the **ACTUAL VALUE** of X , but we are in general not aware of what this is. That is, we have limited information about it. Of course, $AV(s) = s$ for each $s \in S$, that is we assume the members of S are trivially known. The plausible value function PV is mathematically an S -valued function defined on $T \times E_0$, where E is a Boolean algebra of propositions and E_0 denotes the non-zero members of E . We denote by $PV(X|A)$ the value of this function on the pair $(X, A) \in T \times E_0$.

We must make some basic assumptions on how unknown quantities get plausible values. Now, the most basic assumption that can be made which is absolutely obvious from the standpoint of logical consistency is that if our information tells us exactly what value an unknown has, then the plausible value of that unknown given that information must be that value the information is telling us. So we formulate this as our **FIRST AXIOM OF PLAUSIBLE VALUE**.

AXIOM 1. *If T is a set of unknown members of the set S , where S is any set, we assume that $S \subset T$ and AV is a retraction of T onto S . If $X \in T$, if $s \in S$ and the proposition $A \in E_0$ implies that $AV(X) = s$, then $PV(X|A) = s$.*

Notice by Axiom 1 of plausible value, that $PV(\cdot|A)$ for fixed proposition $A \in E_0$ defines a retraction of T onto S , if T is a set of unknown members of S . This is because if $s \in S$, then A trivially implies $AV(s) = s$ so by axiom 1 we have $PV(s|A) = s$.

Our next axiom also makes good common sense from the standpoint of logic. If our information is telling us that two unknowns have the same value, even if we do not know that value, we must choose the same plausible value for both in order to maintain logical consistency.

AXIOM 2. *If $X, Y \in T$ are unknown members of the set S and if the proposition $A \in E_0$ implies that $AV(X) = AV(Y)$, then $PV(X|A) = PV(Y|A)$.*

And now for the examples. Consider a set S and any set D and form the set T of S -valued functions on D , so $T = S^D$. We regard $S \subset S^D = T$ by identifying each member of S with a constant function on D . Let $d \in D$ and define $AV(X) = X(d)$ for each $X \in T$. Of course, taking $PV = AV$ independent of the E_0 variable satisfies the axioms showing consistency.

In particular, consider unknown real numbers. We regard an **UNKNOWN (NUMBER)** as any defined numerical quantity X whose definition tells us it has an exact value but whose definition does not necessarily tell us what that value is. Suppose we have some assumed information in the form of a proposition C which influences our idea of what its value might be. For example, X could be Beethoven's weight in pounds at noon on his fifth birthday. We can take C to be a proposition which states our knowledge of typical weights of five year old children. Clearly 1000 is not a reasonable guess as to what X is, but 45 might not be to far off. As another example, we can take Y to be the current outside temperature in degrees Celsius. If C is the statement of all of our previous knowledge of weather, our experience of the outside air temperature the last time we were outside, as well as what we see by looking out our office window, then we may be able to get a pretty good plausible value of the outside temperature. If we are outside we can probably do even better. Now, our plausible value may be only a guess, and there may be many choices, but we want to imagine that there is some set E of propositions that we will consider and some set T of unknowns that we are interested in, and that for these we choose $PV(X|C)$ for each C in E_0 and each X in T . Now, again, we want to develop the properties of PV based on the idea that as a function on $T \times E_0$ to \mathbb{R} , it must have certain properties to conform to common sense logical consistency.

We can notice that if X and Y are unknown numbers, then we can clearly form $X + Y$ and XY . For instance, X and Y could be the unknowns in the two preceding examples involving weight and temperature. If we have some information about X and Y , then we have information about their sum and product as well. The unknowns have no units in and of themselves, the units are contained in their descriptive information which gives them a numerical value, so any unknown numbers can always be added and multiplied. Since it is reasonable to assume that if we are interested in a pair of unknown numbers we might also need to deal with their sum and product, we assume then that T is closed under the operations of addition and multiplication, making it a **RING**. This is mainly a convenience, and we should point out that for our proof of the Cox Theorem, we only need to assume closure under multiplication of unknowns by indicators, which we proceed to define next. We assume that if A and B belong to E , then so do $A \& B$, the negation of A , denoted $not A$, and A or B and that E is nonempty, so it is a

BOOLEAN ALGEBRA of propositions. If C is a proposition, then we can use it to define an unknown I_C which has the value 1 if C is true and the value 0 if C is false, and which we call the **INDICATOR UNKNOWN** of C . Notice the truth value of a proposition is entirely contained in its indicator unknown, so interest in whether or not a particular proposition is true is equivalent to interest in the value of its indicator unknown. Consequently, we assume that T contains all indicators of propositions in E . As with general sets, we will regard the real scalar field, \mathbb{R} as special unknowns which are known values under any information, (C implies $AV(r) = r$ for every number r), so we assume that T contains \mathbb{R} , the field of real numbers and therefore in particular, T is an **ALGEBRA** over \mathbb{R} . As far as the Boolean algebra E is concerned, we can note that in general, by Stone's Theorem [11], we can embed E as a Boolean algebra of idempotents in the algebra C_E of continuous real valued functions on the Stone space of E . We can therefore regard the algebra T as an algebra over C_E as a way of more concretely thinking of the way indicators act on unknowns. Similarly, if W is a vector space and T is a vector space of unknown members of W , then we can regard the action of indicators on T as coming from a C_E -module structure on T . Thus, if K is any commutative algebra over \mathbb{R} , then we can take any K -module T with \mathbb{R} -submodule W , a retraction AV of T onto W , and for each idempotent A in $K \setminus 0$ choose a retraction $PV(\cdot|A)$ of T onto W , to produce a mathematical model of the setup for unknown vectors in W . Since these retractions can be chosen to be linear, we see that there exist many such setups.

We summarize these comments as our next axiom.

AXIOM 3. *We assume a set T of real unknowns is a commutative algebra with identity over the field of real numbers, \mathbb{R} , and that it contains the indicator unknowns of all propositions in the Boolean algebra of propositions E , that is we assume that the set of indicators of members of E is a Boolean algebra of idempotents in T .*

We want to put order axioms on our plausible numerical values so that plausible numerical values are logically consistent with common sense. In particular, we will take as our next axiom:

AXIOM 4. *If X and Y are in T and if C is in E_0 , and if C implies that $AV(X) \leq AV(Y)$, then $PV(X|C) \leq PV(Y|C)$.*

This axiom merely says that we must choose the ordering of plausible values so as not to contradict the order information we have about the underlying numerical unknowns. As an immediate consequence of this axiom, we have that if C implies that $AV(X) = AV(Y)$, then $PV(X|C) = PV(Y|C)$. This is simply because for real numbers, $=$ is the same as \leq & \geq . Thus, we see that Axiom 2 in the case where $S = \mathbb{R}$, is a consequence of Axiom 4 for the case where $S = \mathbb{R}$. In particular, as a consequence of Axiom 1, if r is any real number, then since C trivially implies $AV(r) = r$, it follows that $PV(r|C) = r$. Thus for fixed C , the plausible value $PV(X|C)$ viewed as a function of X in T is in fact a retraction of T onto $\mathbb{R} \subset T$. Now an immediate consequence of Axioms 1 and 4 is that if a and b are real numbers and C implies that $a \leq X \leq b$, then

$$a \leq PV(X|C) \leq b.$$

If A, C are in E , then $0 \leq I_A \leq 1$, so by Axioms 1 and 4 we can immediately conclude that

$$0 \leq PV(I_A|C) \leq 1.$$

In view of the preceding inequality, we define the **PLAUSIBILITY** of A given C , denoted $PL(A|C)$, by

$$PL(A|C) = PV(I_A|C).$$

Now, it is certainly reasonable that if X is in T and we have determined $PV(X|C)$ and if r is any real number then we should be able to determine $PV(rX|C)$ from r and the purely numerical value $PV(X|C)$. For instance, we should be able to change units and do unit conversions

directly on the plausible values (if you think the plausible value for the outside temperature is 20 degrees Celsius, then you should think it is 68 degrees Fahrenheit). At least we should be able to rescale plausible values under unit changes, even if we do not accept changes of zero point as in temperature conversion. This leads to our next axiom:

AXIOM 5. *If r is any real number, if C is any proposition in E_0 , and if X and Y are unknowns in T , and if $PV(X|C) = PV(Y|C)$, then $PV(rX|C) = PV(rY|C)$. In other words, we assume that $PV(rX|C)$ for fixed $r \in \mathbb{R}$ depends only on $PV(X|C)$.*

Thus, by (1) of corollary [2.1], we have homogeneity of plausible value:

$$(3.1) \quad PV(rX|C) = rPV(X|C)$$

Finally, we consider the axiom that leads to our form of Cox's Theorem which we shall call the **COX AXIOM**:

AXIOM 6. *If A, C are fixed in E , if X_1, X_2 are in T , if $PV(X_1|A\&C) = PV(X_2|A\&C)$, then $PV(X_1I_A|C) = PV(X_2I_A|C)$. That is, we assume that as a function of X , the plausible value $PV(XI_A|C)$ depends only on $PV(X|A\&C)$.*

To motivate this axiom, notice that if A is false, then $XI_A = 0$, whereas if A is true, then we are evaluating the plausible value of X with both A and B being true, which should somehow depend only on $PV(X|A\&C)$. Notice the asymmetry here, which prevents any consideration of the multitude of possibilities in plausibility theory [27]. We cannot put X in the position of the given information, the first variable of PV can only be an unknown and the second variable can only be a statement. Moreover, $PV(XI_A|C)$ cannot depend on the numerical value of $PV(X|C)$ because we could generally have unknowns X and Y with $PV(X|C) \neq PV(Y|C)$ but with A implying that X and Y are equal, in which case we clearly must have that $PV(XI_A|C) = PV(YI_A|C)$. This leads directly to our form of the product rule of Cox's Theorem.

Theorem 3.1. *If X is any unknown number in T and if A, C are any propositions in E , with $A\&C \in E_0$, then*

$$(3.2) \quad PV(XI_A|C) = PV(X|A\&C)PV(I_A|C).$$

Proof. This is an immediate consequence of (4) in corollary (2.1) and the previous axioms, where we take $P_3 = PV(\cdot|C)$, $P_1 = P(\cdot|A\&C)$, and $P_2 = PV(\cdot|C)$. \square

Corollary 3.1. *If A, B, C belong to E , with $B\&C \in E_0$, then*

$$(3.3) \quad PL(A\&B|C) = PL(A|B\&C)PL(B|C).$$

which is the standard product rule of Cox's Theorem.

We need to point out here, that our approach to the Cox theorem (3.1) has eliminated the problems which allow the counterexample of Halpern [8]. We do not need to have an associative multiplication on the real line or an interval, we do not need to assume any continuity or differentiability or divisibility, we do not need to assume that our Boolean algebra of propositions has sufficiently many plausible values to have dense range in an interval of numbers. We do not even need to assume a function of two real variables as Cox does, we merely assume that for fixed $A \in E$ that the plausible value of XI_A as it depends on X is somehow only depending on the plausible value assigned to X , a considerably weakened assumption. In fact, we could have the hypothesis only for a particular A and the result then applies to that particular A . That is, by (2.1), we see that we do not even need to assume this for all $A \in E$ at once, it is enough to assume it for a single $A \in E$ and to assume the homogeneity of that single

indicator. In effect, by passing to unknowns and using indicator functions, the rescalability of plausibilities (encoded in the homogeneity of plausible value) causes the range to be the whole real line and consequently, the universal function assumed by Cox will here have to have domain $\mathbb{R} \times \text{Im}(PL)$. It is thus the homogeneity, not the additivity assumed by some authors [15], which is the crucial ingredient which gives the result. Also, we have assumed our set of unknowns forms an algebra with identity over the reals as it seems most natural, but we really only used the fact that we have a set of unknowns that is closed under scalar multiplication and contains all indicators from E and the real numbers themselves as a subset. We can also point out that if we drop the axiom of homogeneity (5), then by (3) of the corollary (2.1), we would still obtain a weakened form of the Cox Theorem as a consequence of the other axioms.

An alternative to the Cox Axiom, due to Savage [23], in case of plausibility, is the **SURE THING AXIOM**:

AXIOM 7. *If $X, Y \in T$ and $A, B, A \& B, B \setminus A \in E_0$, and if both $PV(X|A \& B) = PV(Y|A \& B)$ and $PV(X|B \setminus A) = PV(Y|B \setminus A)$, then $PV(X|B) = PV(Y|B)$.*

If we form $Y = PV(X|A \& B)I_A \in T$, then the sure thing axiom implies that $PV(XI_A|B) = PV(Y|B)$, since $PV(XI_A|A \& B) = PV(X|A \& B)$ by axiom 2, and the product rule of Cox's Theorem is then an immediate consequence of this equality and homogeneity from axiom (5).

So far, nothing has been said about additivity of PV . Of course, (2) of corollary (2.1) gives additivity if we assume there is an appropriate general dependence.

Proposition 3.1. *Suppose that S is a set with binary operation, $+$, and T is a set of unknowns of S , which is closed under $+$, and with $S \subset T$. If we assume that $PV(X+Y|A)$ for all unknowns X and Y in T depends only on the values $PV(X|A)$ and $PV(Y|A)$, then*

$$(3.4) \quad PV(X + Y|A) = PV(X|A) + PV(Y|A).$$

Proof. This is an immediate consequence of (2) in corollary [2.1] on taking $P_1 = P_2 = P_3 = PV(\cdot|A)$. \square

What we see here is that the additivity of plausible value in the most general sense possible would be a consequence of the basic logical consistency of meaning together with the mere assumption that some form of law of combination exists. Thus, the same would apply if we were considering plausible values of unknown vectors in vector spaces—if we assume the plausible value of the sum somehow depends on the plausible value of the summands, then the only possible rule is the standard sum rule. Such general additivity laws are usually easy to motivate with examples, or in the case of $S = \mathbb{R}$ by thinking in terms of money, but in the end, whatever the motivation, it includes the motivation that an actual rule exists, and that is already enough. Moreover, the final arbiter on such an assumption has to be whether experience with its use leads to reasonable results. For, notice that if the operation is taken to be ordinary multiplication with $S = \mathbb{R}$, the same argument applies but then the rule is not generally true even for ordinary expectations in ordinary probability theory, which means that for general expectations in probability theory there can be no general rule for getting the expected value of a product from the expected values of the factors. We see from proposition (3.4) that if we axiomatically assume there is some form of rule giving the plausible value of a sum in terms of the plausible values of the individual summands, then the only possible rule is the ordinary sum rule. But, before going that far, let us reconsider the temperature example.

Suppose that X is the outside temperature in degrees Celsius. If our information leads to a best guess of c as the most plausible value, then consistency requires that in degrees Fahrenheit the plausible value is $32 + (9/5)c$. This includes a change in zero point. Thus, consistency with the most general changes of units for any unknown leads to the next axiom:

AXIOM 8. *If a, b belong to \mathbb{R} , if X belongs to T , and if C belongs to E , then*

$$(3.5) \quad PV(aX + b|C) = aPV(X|C) + b.$$

Notice that this axiom implies axioms 1 and 5 and includes a limited form of additivity. Thus, in particular, axiom 6 and this axiom imply the sum rule of Cox's theorem. However, this last axiom allows us to immediately arrive at the properties of plausibility for statements. Because we have

$$\begin{aligned} I_{notA} &= 1 - I_A \\ I_{A\&B} &= I_A I_B \end{aligned}$$

and therefore by de Morgan's Law

$$I_{AorB} = I_A + I_B - I_A I_B.$$

So,

$$PV(I_{notA}) = 1 - PV(I_A),$$

and it is well known [13] that the sum rule of Cox's theorem and the preceding complementation property imply by deMorgan's Law that $PV(\cdot|C)$ is additive on indicators of exclusive propositions. We thus arrive at the usual rules of probability on defining the probability, $P(A|B)$, of A given B by $P(A|B) = PV(I_A|B)$. To obtain the general additivity of plausible value, we now only need to assume the following simpler axiom.

AXIOM 9. *For T an algebra of real unknown numbers, for each fixed $Y \in T$ and $A \in E_0$, the plausible value $PV(X + Y|A)$ depends only on $PV(X|A)$.*

Proposition 3.2. *If T is an algebra of unknown numbers and $X, Y \in T$ with $A \in E_0$, then assuming axioms 8 and 9,*

$$(3.6) \quad PV(X + Y|A) = PV(X|A) + PV(Y|A).$$

Proof. Fix $Y \in T$ and $A \in E_0$. Now, by assumption, on considering $PV(X + Y|A)$ as a function of X alone, the Axiom 9 guarantees a function $f_{(A,Y)}$ satisfying $f_{(A,Y)}(PV(X|A)) = PV(X + Y|A)$, for every $X \in T$. If we take the special case of $X = r \in \mathbb{R}$, then, as $PV(r|A) = r$, and as by Axiom 8 we have $PV(r + Y|A) = r + PV(Y|A)$, it follows that

$$f_{(A,Y)}(r) = PV(r + Y|A) = r + PV(Y|A),$$

for every real number r , and this gives the result. \square

We are of the opinion that the most economical approach to probability theory is to take as axioms, 4, 6, and 8, as these three axioms easily give the Cox Theorem and the rules of probability without having to modify the plausibility function. In addition, merely adding the axiom 9, then gives the full theory of expectation for random variables as well as general unknown numbers. In fact, if we go to complex unknowns, with obvious complex versions of the axioms, and assume that the unknowns form a C^* -algebra, as specifying a PV is equivalent to giving a state, it is known that every state is a bounded linear map [4], so that the usual analysis with measure theory follows from the representation of bounded linear functions as integration with respect to a finite measure.

Suppose that more generally we have a vector space W and we are interested in plausible values for members of a set T of unknown members of W . Then, the obvious modification of the axioms 5 and 6 leads to the conclusion that if X is in T and A, C belong to E , then by (4) of corollary (2.1) we find the obvious generalization of the Cox Theorem again. In fact, we can replace W by a general module M over any possibly noncommutative ring R , and with the

obvious modification of the axioms, we obtain the obvious generalization of the Cox Theorem, where we simply replace indicators by idempotents in ring R and assume in addition that A and C commute as idempotents in R so as to make their product again idempotent. The main point here is that if X is an unknown member of R and v is in W , then we must assume that $PV(Xv|e)$ depends only on $PV(X|e)$ as a function of X keeping e and v fixed. As a consequence of this assumption we find the general rule

$$PV(Xv|e) = [PV(X|e)]v,$$

which combined with (3) in Corollary (2.1) gives the general multiplication rule:

$$PV(e_1Y|e_2) = PV(Y|e_1e_2)PV(e_1|e_2),$$

as long as e_1, e_2 and e_1e_2 , are all idempotents, which is the case if the two idempotents commute. Here, Y is an unknown vector, so we must keep in mind that $PV(Y|e)$ is a member of W whereas $PV(X|e)$ is a member of R . Finally, if we take T to be a C^* -algebra and R to be C^* -subalgebra and P a retraction of T on R , then Corollary (2.1) gives us simple natural conditions for P to be a conditional expectation in C^* -algebra theory, that is conditions for P to be an R -linear map.

Of course, we can produce examples of PV functions by taking in particular function algebras or even noncommutative C^* -algebras. In particular, it is known that if T is a C^* -algebra with identity, and if we take for our set of unknowns the set S of self-adjoint members of T , then any state of the C^* -algebra restricted to S will serve as a consistent way of assigning plausible values which in fact satisfy the general additivity of proposition (3.4). In fact, if T is any separable C^* -algebra, we can take the universal representation and produce a state, f , which will not vanish on any nonzero positive element. We then define the plausible value $PV(X|A) = f(XA)/f(A)$, for any $X, A \in T$ such that A is a nonzero idempotent. In particular, if T is commutative, then we know that the states which are multiplicative are exactly the pure states, which are the point evaluations under any representation of such an algebra as an algebra of continuous functions on a compact Hausdorff space. Thus, the assumption that plausible value is generally additive is a reasonable assumption, whereas we see that the additional assumption of multiplicativity would be too restrictive. In general, it is known from Choquet theory that the set of all states of a C^* -algebra is a compact convex subset of the continuous dual of the algebra under the weak*-topology, and that it is the closed convex hull of the pure states, as these form the set of extreme points of that convex set [4].

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