

**THE FULLY COVARIANT ENERGY
MOMENTUM STRESS TENSOR OF THE GRAVITATIONAL FIELD AND
THE EINSTEIN EQUATION FOR GRAVITY IN GENERAL RELATIVITY**

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4 March 2009

ABSTRACT. We give a fully covariant energy momentum stress tensor for the gravitational field which is easily physically and intuitively motivated, and which leads to a very general derivation of the Einstein equation for gravity. We do not need to assume any property of the source matter fields' energy momentum stress tensor other than symmetry. We give a physical motivation for this choice using laser light pressure. As a consequence of our derivation, the energy momentum stress tensor for the total source matter fields must be divergence free, when spacetime is 4 dimensional. Moreover, if the total source matter fields are assumed to be divergence free, then either the spacetime is of dimension 4 or the spacetime has constant scalar curvature.

Mathematics Subject Classification (2000) : 83C05, 83C40, 83C99.

Keywords : Gravity, general relativity, Einstein equation, energy density.

1. INTRODUCTION

Our purpose here is two-fold. First, we wish to give a fully covariant energy momentum stress tensor for the gravitational field. Second, we will use our gravitational field energy momentum stress tensor to give a general derivation of the Einstein equation for gravity, and find as a consequence that the divergence of the energy momentum stress tensor for matter and fields other than gravity must be zero.

This manuscript is an expanded version of a manuscript submitted for publication. In communication with physicists, it has come to my attention that in general, they are not as comfortable as mathematicians with some of the more modern results in analysis and differential topology and geometry. Here we will attach an appendix for each mathematical topic that possibly needs more coverage.

Since Einstein's and Hilbert's original "derivations" of the Einstein equation for gravity in classical general relativity (CGR), there have appeared too many different derivations to list. The many different types of derivations are summarized in [47]. In fact, all these subsequent derivations as well as Hilbert's original derivation contrast markedly from Einstein's original derivation and usually appeal to some abstract mathematical principle which though desirable, is usually not justifiable beyond mere desire. For instance, one of the most popular textbook derivations simply modifies one side of the equation to make it have zero divergence on grounds that physical considerations make the other side, the matter energy momentum stress tensor, have zero divergence.

If one uses a Lagrangian or variational method, then one is immediately faced with the question of justifying the choice of Lagrangian which is generally not really possible. In fact, to quote from [25], after a detailed rigorous treatment of the Lagrangian formulation of Einstein's equation (pages 271-279), "That the Lagrangian ansatz described in this section works is by no means trivial and I have no explanation for it".

On the other hand, in Einstein's original derivation, [17], we see the realization that mathematically the Ricci tensor should be proportional to the source which should be the total energy density due to both the energy-stress tensors of matter as well as the gravitational field itself. However, in [17], Einstein was not able to arrive at a fully covariant tensor expression for the energy density of the gravitational field. Instead, using

a Hamiltonian or variational method (therefore a weakness in the argument) he arrived at a pseudo-tensor defined in terms of the connection coefficients for certain special coordinate systems and which he argued (on grounds it could be shown that the pseudo-tensor was coordinate divergence free) served to give the energy density of the gravitational field for purposes of deriving the equation.

As the arguments in [16] leading up to the development of CGR show, Einstein was clearly thinking of the energy of the gravitational field in a Newtonian way, since in particular, the connection coefficients are the generalized gravitational forces from the Newtonian viewpoint. In particular, in his elementary analysis of the conversion of gravitational potential to energy through absorption of a light pulse, he represented the gravitational potential as height. Moreover, in [17], Einstein was very clear that his equation was using the energy density of the gravitational field in addition to the energy-stress tensor as the total source of gravity. Indeed, his derivation there uses a break-up of the already accepted vacuum equation as an equation for the gravitational field pseudo-tensor and then he merely argued that the source should have the matter tensor added in with the pseudo-tensor so that putting the pseudo-tensor back to the Ricci side of the equation gave the final fully covariant field equation.

In fact, subsequent attempts to mathematically characterize the energy of the gravitational field have all basically clung to the Newtonian framework which makes the energy of the gravitational field a function of a non-local arrangement of masses and energies, or combinations of connection coefficients, the results all giving pseudo-tensors. So much so, that these views are now taken for granted to the point that in [31] we have the claim of the impossibility of existence of a local energy density tensor for the gravitational field (see also [49],[22],[19],[6],[48]). This attitude clearly persists to the present as expressed, for instance, in chapter 3 of [48], or [37].

The non-localizability of gravitational field energy is often, but unnecessarily, used as a justification for the development of the profusion of mathematically inspired notions of quasi-local mass, which all have their advantages and drawbacks as discussed in [48] and [30], along with extremely involved analysis required to arrive at their basic properties. In the case of pseudo-tensors and quasi-local mass definitions developed using Hamiltonian methods, the results are sensitive to boundary conditions, as pointed out in [37]. As these attempts at forming quasi-local mass are potentially very valuable for the global analysis of general relativistic models, the question of the actual energy density is often irrelevant.

As energy in physics has historically been defined to be a conserved quantity, the extension of notions of energy in general relativity have been heavily influenced by this desirable property. But as soon as Einstein formulated $E = mc^2$, energy in relativity took on a physical reality beyond a mere calculation tool, as it was in Newtonian physics. This means that any physically real energy, and in particular, gravitational energy, must be physical, localizable, and itself a source of gravity. For instance, to quote H. Bondi [3], "In relativity a non-localizable form of energy is inadmissible, because any form of energy contributes to gravitation and so its location can in principle be found." Whether or not it is conserved then becomes a separate question. Unfortunately we will find this must be generally answered negatively, unless the only matter fields present have energy momentum stress tensor having zero contraction, as is the case for the electromagnetic field energy momentum stress tensor. Trying to force it to be conserved can only lead to problems. In fact Dirac, the ultimate mathematical physicist, puts it best, concluding that for the gravitational field energy, being localizable and being conserved (meaning divergence free) are not mutually compatible (page 62 in [10]). Consequently, we must be content to think of the various pseudo-tensors which are conserved in certain situations as useful to the extent they are helpful in calculations, but we should not think of them as giving actual real inertial energy. Likewise, we must be content to think of the various definitions of quasi-local mass or energy as tools for calculation and thus judge them purely on their utility for helping us understand global solutions to the Einstein equation.

In summary, we give a physical motivation for the postulate that each observer should view the sum of principal pressures as being the energy density of the gravitational field he observes. We demonstrate that mathematically, Einstein's equation is equivalent to the combination of three statements (see Theorems 6.1 and 7.1). The first is that each observer should see Newton's Law for infinitesimal tidal acceleration at his location, in a manner to be made mathematically precise. The second is that each observer must include the gravitational energy density he observes in the source. The third is that each observer must see the

energy density of the gravitational field as the sum of principal pressures. The first two statements are obviously the minimal required modifications of Newton's Law to give a law which makes sense in relativity, and the third is the postulate which we motivate physically with an argument involving lasers. The fact that these assumptions are mathematically equivalent to Einstein's equation for gravity would seem to make our postulate for the energy density of the gravitational field very compelling. We prove a mathematical theorem we call the observer principle which is really a special case of the uniqueness of power series for analytic functions which is at the heart of the principle of analytic continuation. As a consequence of the observer principle, our postulate that each observer sees the gravitational field as the sum of principal pressures means that the covariant energy momentum stress tensor for the gravitational field (see Theorem 6.1) is $T - c(T)g$ where $c(T)$ denotes the contraction of T .

In order to make the presentation clear to a more general audience than specialists in the field, we have included possibly more details than an expert will need. Since the dimension of spacetime does not really enter into the argument, we will actually derive the gravitation equation for a spacetime of $n + 1$ dimensions, and arrive at the usual Einstein equation in case $n + 1 = 4$. It is in higher dimensions that the weakness of pure mathematical arguments involving desirable forms of equations or of purely Lagrangian-variational methods becomes clear. It gives the Einstein tensor as the geometric side of the equation plus other terms with free parameters [29]. No clearly unique equation emerges. On the other hand, our energy momentum stress tensor for the gravitational field dictates a clear choice for the gravitation equation in higher dimensions. In particular, the resulting general gravitation equation shows the assumption of infinitesimal conservation (zero divergence) of the energy momentum stress tensor of the matter and fields other than gravity implies that the spacetime must have constant scalar curvature in spacetime dimension other than $n + 1 = 4$. This would seem to be a strong physical indicator that spacetime should be, or at least appear to be, 4-dimensional. Thus it is only for spacetime of dimension $n + 1 = 4$ that our derivation gives $div(T) = 0$ as an automatic consequence, where T denotes the energy momentum stress tensor of all matter and fields other than gravity.

2. MATHEMATICS AND THE OBSERVER PRINCIPLE

For general references on differential geometry, semi-Riemannian and Lorentz geometry, we refer to [27], [25], [31], [49], and [38]. To begin, we assume that our spacetime is an $(n + 1)$ -manifold M equipped with a Lorentz metric tensor g , with signature $(-, +, +, +, \dots, +)$, and we denote by ∇ the resulting Levi-Civita Koszul connection or covariant differentiation operator on M . We use TM for the tangent bundle of M and $T_m M$ to denote the tangent space of M at $m \in M$. If $f : M \rightarrow N$ is a differentiable map of manifolds, say of class C^r , then $Tf : TM \rightarrow TN$ is the tangent map which is of class C^{r-1} and we note here the simple property $T(hf) = (Th)(Tf)$ as regards composition of differentiable mappings. In case $f(m) = n \in N$, then $T_m f : T_m M \rightarrow T_n N$ is a linear map. It is convenient in this setting to refer to $u \in T_m M$ as a unit vector to mean merely $|g(u, u)| = 1$. Thus u is a time-like unit vector when $g(u, u) = -1$. Because M is a Lorentz manifold, each of its tangent spaces is a Lorentz vector space of dimension $n + 1$.

If u is a time-like unit vector in a Lorentz vector space, L , then $u^\perp \subset L$ is a Euclidean space. We define the projection operator $P_u : u^\perp \rightarrow u^\perp$ by $P_u(v) = v + g(v, u)u$, for any $v \in L$. If B and C are any linear transformations of L , we say that C is the adjoint of B to mean that $g(Bv, w) = g(v, Cw)$ for all pairs of vectors $v, w \in L$. In this case, C is uniquely determined by B and we write $C = B^*$. It is easy to see that in general for any two linear operators B and C on L we have $(BC)^* = C^*B^*$. In particular, P_u is self-adjoint, $P_u^* = P_u$ and as well $P_u^2 = P_u$, so P_u is an idempotent in the algebra of linear maps of L .

If B is any self-adjoint linear transformation of L , then $P_u B P_u$ is also self-adjoint but has u^\perp as an invariant subspace and therefore defines a self-adjoint linear transformation $B_u : u^\perp \rightarrow u^\perp$. But since u^\perp is a Euclidean space, this means that B_u is diagonalizable. We call the eigenvalues (also called proper values) of B_u the u -spatial eigenvalues of B , we call the principal axes or lines through eigenvectors of B_u the u -spatial principal directions of B , and we call the average of the eigenvalues of B_u the u -isotropic eigenvalue of B . Thus, if λ_u is the u -isotropic eigenvalue of B , then $trace(B_u) = n\lambda_u$. In particular, if u is also an eigenvector of B with eigenvalue r , then B is completely diagonalizable and $trace(B) = r + n\lambda_u$. But, more generally, since $g(u, u) = -1$, we always have,

$$(2.1) \quad \text{trace}(B) = -g(Bu, u) + n\lambda_u,$$

even if u is not an eigenvector of B . Thus, we emphasize that even though B_u is always diagonalizable, B itself need not be.

Using the time-like unit vector u allows us to also define a Euclidean metric or inner product g_u on L by defining $g_u(v, w) = g(v, w) + 2g(v, u)g(u, w)$. This makes L a topological vector space and in case L is finite dimensional, this gives L its unique vector topology. Thus even though the Euclidean inner product on L depends on the choice of u , the resulting topology does not. It is easy to see that for $B = B^*$ to be also self-adjoint with respect to the Euclidean inner product g_u , it is necessary and sufficient that u be an eigenvector of B in which case B is itself then diagonalizable.

If T is a second rank tensor on L , which is merely to say that T is a real-valued bilinear map on L , then there is a unique linear map $B_T : L \rightarrow L$ with $T(v, w) = g(B_T v, w)$, for all $v, w \in L$. We can now invariantly define the *contraction* of T , denoted $c(T)$, by

$$(2.2) \quad c(T) = \text{trace}(B_T).$$

Any question of eigenvalues, eigenvectors, or diagonalizability for T is really the same question for B_T . Clearly to say T is symmetric is the same as saying that B_T is self-adjoint. Thus for T symmetric, its u -spatial principal directions are those of B_T , its u -spatial eigenvalues are those of B_T and its u -isotropic eigenvalue is that of B_T . It is customary to call the u -isotropic eigenvalue of T the *isotropic pressure* for the observer with velocity u in case T is an energy momentum stress tensor and $L = T_m M$, and we will denote this by p_u , in this case. Thus for this situation we have, by (2.1) and (2.2),

$$(2.3) \quad c(T) = \text{trace}(B_T) = -T(u, u) + np_u.$$

Also, in this situation, $T(u, u)$ is always designated as the *energy density* observed by the observer with velocity u .

For $v \in T_m M$, we denote by ∇_v the covariant differentiation operator along v at m . We have then the Riemann curvature operator, \mathcal{R} , given by

$$(2.4) \quad \mathcal{R}(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u, v]},$$

where u and v are any tangent vector fields on an open subset U of M . We note that $\mathcal{R}(u, v)$ actually defines a vector bundle map of the tangent bundle $TM|U$ to itself covering the identity map of U , and it as well then determines the Riemann curvature tensor, *Riemann*, of fourth rank, which means that \mathcal{R} is itself an alternating second rank tensor field on M which at each point $m \in M$ gives a linear transformation valued tensor on $T_m M$. In particular, this means that $\mathcal{R}(u, v)$ is defined, giving a linear transformation of $T_m M$ for any pair of tangent vectors $u, v \in T_m M$. One of our main concerns is the certain contraction of *Riemann* known as the Ricci tensor, *Ric*. In fact in any frame at $m \in M$ with basis (e_α) for $T_m M$ and dual basis (ω^α) , we have, using the summation convention,

$$(2.5) \quad \text{Ric}(u, v) = \omega^\alpha(\mathcal{R}(e_\alpha, u)v), \quad u, v \in T_m M.$$

Our notation is chosen to emphasize we are not restricting ourselves to coordinate frames nor to orthonormal frames unless explicitly stated. Thus, we will refrain from using the abstract index notation, as it is too often restricted to imply coordinate framing. In particular, for any pair of tangent vectors $v, w \in T_m M$, the curvature operator defines another linear transformation $\mathcal{K}(v, w)$ of $T_m M$ defined by

$$(2.6) \quad \mathcal{K}(v, w)z = \mathcal{R}(z, v)w, \quad z \in T_m M.$$

Then, among the many basic symmetries of the curvature tensor, one is immediately equivalent to

$$(2.7) \quad \mathcal{K}(v, w)^* = \mathcal{K}(w, v), \quad v, w \in T_m M,$$

and as

$$(2.8) \quad Ric(v, w) = trace[\mathcal{K}(v, w)],$$

the symmetry of Ric then follows immediately from (2.7). Moreover, if u is any tangent vector, then $\mathcal{K}(u, u)$ is self-adjoint or symmetric, clearly vanishes on the line through u , and therefore has u^\perp as an invariant subspace. Thus $\mathcal{K}(u, u)$ really "lives" on u^\perp , the orthogonal complement of u in $T_m M$. We shall denote by $A_u^{(geo)}$ the restriction of $-\mathcal{K}(u, u)$ to u^\perp , so $A_u^{(geo)} : u^\perp \rightarrow u^\perp$ is a self-adjoint linear transformation of the Euclidean space u^\perp , in the case that u is a time-like unit vector. Thus for this case that u is a time-like unit vector, the metric tensor is positive definite on this orthogonal complement, and it follows that $Ric(u, u)$ is simply the sum of the eigenvalues of $-A_u$ or of $\mathcal{K}(u, u)$. In general, if $(u, e_1, e_2, \dots, e_n)$ is an orthonormal frame with u a time-like unit vector, then we note that

$$(2.9) \quad g(e_k, \mathcal{R}(e_k, u)u) = g(e_k, \mathcal{K}(u, u)e_k)$$

is the negative of the Riemann sectional curvature of the span of u and e_k , in $T_m M$, because $g(u, u) = -1$. Thus, the eigenvalues of $A_u^{(geo)}$ are the principal Riemann sectional curvatures through u . We can now symmetrize and define

$$(2.10) \quad S(v, w) = Sym(\mathcal{K}(v, w)) = \frac{1}{2}[\mathcal{K}(v, w) + \mathcal{K}(w, v)], \quad v, w \in T_m M.$$

We see immediately from (2.7) that S is a symmetric linear transformation valued tensor whose values are themselves self-adjoint transformations of $T_m M$. Moreover, we also have $S(v, v) = \mathcal{K}(v, v)$ for each $v \in T_m M$, whereas, $Ric(v, w) = trace S(v, w)$, for any $v, w \in T_m M$.

Now, mathematically, $T_m M$ is a Lorentz vector space of dimension $n+1$, so as above, taking any time-like unit vector, say u , and defining $g_u(v, w) = 2g(u, v)g(u, w) + g(v, w)$ gives a Euclidean metric on $T_m M$ making it in particular into a Banach space of finite dimension. Thus, $T_m M$ is an example of a Banachable space—a topological vector space whose topology can be defined by a norm. This topology is actually independent of the choice of u in case of finite dimensions. Differential geometry can be easily based on such spaces, and for some examples in infinite dimension, the interested reader can see [26], [5], [14] and [15]. In particular, the theory of analytic functions and power series all goes through for general Banachable spaces. We would like to point out how this can be applied to the theory of Lorentz vector spaces. Suppose that E and F are Banachable spaces and S is a continuous symmetric multilinear map (tensor) on E with values in F , of rank r . We can define the monomial function $f_S : E \rightarrow F$ by the rule $f_S(x) = S(x, x, x, \dots, x) = Sx^{(r)}$, and then f_S is an analytic function. In fact, if $x_1, x_2, x_3, \dots, x_r \in E$, then differentiating, using proposition 3.3 and repeated application of propositions 3.5 and 3.8 of [27], page 10, we find

$$(2.11) \quad D_{x_1} D_{x_2} D_{x_3} \dots D_{x_r} f_S(a) = (n!) S(x_1, x_2, x_3, \dots, x_r), \quad a \in E.$$

From (2.11), we see very generally that if U is any open subset of E on which f_S is constant, then in fact, $S = 0$, since we can choose $a \in U$. Indeed, if $a \in U$, since f_S is constant on U , it follows that the derivative on the left side of the equation (2.11) is 0, and hence also the right side, for every possible choice of vectors $x_1, x_2, x_3, \dots, x_r \in E$. But notice that a does not appear on the right hand side of (2.11), only $S(x_1, x_2, x_3, \dots, x_r)$, and the vectors $x_1, x_2, x_3, \dots, x_r$ can be chosen arbitrarily. Thus, $S = 0$ follows. This is just a very special case of the principle of analytic continuation. We have therefore proven the following mathematical theorem.

Theorem 2.1. ANALYTIC CONTINUATION. *If A and B are both symmetric tensors of the same rank, r , on E with values in F and if $Av^{(r)} = Bv^{(r)}$ for all v in the nonempty open subset U of E , then $A = B$.*

We have labelled this as analytic continuation, as it is a well known special case of the uniqueness of general power series (there is only one term here). For a purely algebraic proof in the case $r = 2$, which is the case of most importance here, we refer the interested reader to [12]. See also page 72 of [47] for a proof using differentiation for the case $r = 2$ which is similar in form to that given here next.

Corollary 2.1. OBSERVER PRINCIPLE. *If A and B are both symmetric tensors of rank r on T_mM with values in F , and if $Au^{(r)} = Bu^{(r)}$ for every time-like unit vector in T_mM , then $A = B$.*

Proof. Since f_A and f_B are homogeneous functions of degree r , it follows that the hypothesis guarantees $Av^{(r)} = Bv^{(r)}$ for all v in the light cone of T_mM which is an open subset of T_mM . \square

Of course, if we define $U(T_mM)$ to be the set of time-like unit vectors in T_mM , then this set has a topology called the relative topology as a subset of T_mM and we have a retraction function given by normalization which retracts the light cone onto $U(T_mM)$. It follows immediately that if W is any (relatively) open subset of $U(T_mM)$, then the hypothesis of the observer principle can be weakened to merely require $Au^{(r)} = Bu^{(r)}$ for each $u \in W$. In particular, if we choose a time orientation on T_mM , then we can merely require $Au^{(r)} = Bu^{(r)}$ for each future time-like unit vector in T_mM . This is in a sense, the essence of the **Principle of Relativity**, for instance, as applied to second rank symmetric tensors—a law (at m), say $A = B$, should be true for all observers (at m) and conversely, if true for all observers (at m), that is if $A(u, u) = B(u, u)$ for all (future) time-like unit vectors $u \in T_mM$, then it should be a law (at m) that $A = B$.

We have stated the *observer principle* as corollary to the special case of the mathematical principle of analytic continuation to emphasize the fact to the casual reader that it is really a theorem in pure mathematics, and as such, its proof is completely rigorous. We call it the observer principle merely to emphasize how it will be used in what follows.

Notice that the observer principle can be applied to S of (2.10) as well as to Ric , as tensors on T_mM . Thus, the observer principle says in a sense that these symmetric tensors are *observable*, in the sense that they are completely determined at a given event by knowing how all observers at the event see their monomial forms.

At this point we want to remark that if E is any vector space with a positive definite inner product, g , and if $A : E \rightarrow E$ is any linear transformation of E , then A can be viewed as a vector field on E , say \mathbf{v}_A where $\mathbf{v}_A(x) = A(x)$ for $x \in E$, and as well it defines the dual 1-form λ_A on the Riemannian manifold E defined by $\lambda_A(x)(w) = g(A(x), w)$. We record the following result as a proposition for future use. Its proof is an easy exercise.

Proposition 2.1. *For the linear transformation $A : E \rightarrow E$ of the Euclidean space E , we have*

$$(2.12) \quad \operatorname{div}_E \mathbf{v}_A(0) = (\operatorname{div}_E A)(0) = \operatorname{trace}(A),$$

where div_E denotes the ordinary divergence operator on vector fields defined on E . Moreover, λ_A is a closed 1-form (meaning $d\lambda_A = 0$) if and only if A is self-adjoint as a linear transformation of E .

For M a Lorentz manifold, $m \in M$, and $u \in T_mM$ a time-like unit vector, we will call u an observer at m . We set $E(u, m) = u^\perp \subset T_mM$ and call $E(u, m)$ the observer's Euclidean space (at m). Choose an open subset W of T_mM , with $0 \in W$ and make the choice small enough that the exponential map carries W diffeomorphically onto a geodesically convex ([25], page 131) open subset W_L of M containing m . Denote the image of $W_E = W \cap E(u, m)$ under this exponential diffeomorphism by W_R . We shall call W_R the observer's Riemannian space (at m), whereas we refer to W_E as the observer's Euclidean neighborhood. Thus, we should intuitively think of W_E as the Euclidean space an observer thinks he is in if he is unaware of curvature, whereas W_R is the space the sophisticated observer thinks he is in when he is aware of curvature. Any linear transformation, A of $E(u, m)$, and in particular, $A_u^{(geo)}$, can be viewed by the observer as a vector field \mathbf{w} on his Euclidean space, and by (2.12), the divergence, $\operatorname{div}_E \mathbf{w}(0)$ is simply the trace of A , where $E = E(u, m)$.

3. THE RICCI TENSOR AND SPATIAL DIVERGENCE

In order to see how the Ricci tensor enters into the theory of gravity, we should recall the equation of geodesic deviation. If $[-a, a]$ and $[-b, b]$ is a pair of intervals in \mathbb{R} , then a Jacobi field is a smooth map $J : [-a, a] \times [-b, b] \rightarrow M$ such that for each fixed $\sigma \in [-b, b]$ the map $J_\sigma : [-a, a] \rightarrow M$, given by $J_\sigma(\tau) = J(\tau, \sigma)$, is a unit speed geodesic in M . We can then form local vector fields u, s on an open neighborhood of the image of J in M , denoted $Im J$, so that

$$(3.1) \quad u(J(\tau, \sigma)) = \partial_\tau J(\tau, \sigma), \quad s(J(\tau, \sigma)) = \partial_\sigma J(\tau, \sigma).$$

Thus we must have $[s, u] = 0$ and $\nabla_u u = 0$, on $Im J$, so we find

$$(3.2) \quad \mathcal{R}(s, u)u = -\nabla_u \nabla_s u.$$

We will call s in this situation a tangent Jacobi field along J_0 , and at each point it gives the infinitesimal separation vector. In fact, given m a point on J_0 and any unit vector $s_m \in T_m$ which is orthogonal to $u(m)$, we can arrange that $s(m) = s_m$.

Since our connection is assumed to be the unique torsion free metric connection, we have

$$[s, u] = \nabla_s u - \nabla_u s,$$

so the condition that $[s, u] = 0$ gives $\nabla_s u = \nabla_u s$ in our present case. In view of (3.2), we then find the equation of geodesic deviation on $Im J$,

$$(3.3) \quad \mathcal{K}(u, u)s + \nabla_u \nabla_s u = \mathcal{R}(s, u)u + \nabla_u \nabla_s u = 0.$$

In other words, ∇_u^2 "is" the quadratic form of $-\mathcal{K}$ or $-S$ applied to u , so through $-\mathcal{K}(u(m), u(m))$ we see $A_{u(m)}^{(geo)}$ is the linear transformation of $E(u, m)$ giving the infinitesimal tidal acceleration field, $\mathbf{a}_{u(m)} = A_{u(m)}^{(geo)}$ at m , a vector field on $E(u, m)$ defined by $\mathbf{a}_{u(m)}(s) = A_{u(m)}^{(geo)}s$, for any separation vector $s \in E(u, m)$.

Of interest to operator theorists here (see [11] for spectral theory and functional calculus of operator fields) could be the observation that in some sense we have found a relationship between $\nabla_{u(m)}$ and $(A_{u(m)}^{(geo)})^{1/2}$.

Now, we simply combine the little proposition (2.1) together with (2.8), and find at m , with 0_m denoting the zero vector of $T_m M$,

$$(3.4) \quad Ric(u(m), u(m)) = trace(\mathcal{K}(u(m), u(m))) = -trace(A_{u(m)}^{(geo)}) = -div_{E(u, m)} A_{u(m)}^{(geo)}(0_m).$$

We are interpreting this result as relating to the observer's flat Euclidean space divergence of his flat Euclidean space infinitesimal tidal acceleration field. Moreover, by (2.7) $A_{u(m)}^{(geo)}$ is a self-adjoint linear transformation of the Euclidean space $E(u, m)$.

For comparison, we point out that our discussion above for (3.4) is also the content of results in [38], pages 225-219 and 8.9, page 219, as well as [47], 4.2.2, page 114. We can notice here that by the observer principle, knowledge of $\mathbf{a}_{u(m)} = A_{u(m)}^{(geo)}$ for every possible (even just future pointing) time-like unit vector $u \in T_m M$ would, by (3.3) and (2.9), determine $S = Sym(\mathcal{K})$ at m and thus all Riemann sectional curvatures which, as is well known in differential geometry (see for instance [38], page 79), in turn determines the entire Riemann curvature tensor at m , that is both the Ricci curvature and the contraction free part known as the Weyl curvature (see [49] or [22] for its definition), as pointed out in [49], pages 41-53.

Because the Weyl curvature is contraction free (all its contractions are zero), this means the result (3.4) will only depend on the Ricci tensor and not the Weyl curvature. Thus, even though the actual tidal acceleration field as a vector field on W_R would have derivatives in general depending on the Weyl curvature, the particular combination of derivatives we form, the observer's flat Euclidean space divergence of the infinitesimal tidal acceleration, will necessarily be independent of the Weyl curvature. Of course, if the divergence of the vector field $\nabla_u^2 s$ defined on W_R , the observer's Riemannian space, is calculated, then

the full Riemann tensor enters in and there seems to be no simple symmetric second rank tensor whose monomial form will give the Riemannian divergence of the tidal acceleration field on W_R , and moreover, the Weyl tensor will enter into the result. Worse yet, as a function of the separation vector at the given event $m \in M$, the tidal acceleration in W_R and its divergence would be a complicated function which would not have the tensor property as it depends on how the separation vector is extended to be a vector field in the neighborhood of the event m .

At this point, one might object that the observer could be rotating which would introduce fictional acceleration into $\mathbf{a}_{u(m)}$, and that is correct. A more sophisticated analysis here could deal with this purely mathematically (see for instance [19], [31], or [47]), but let us allow that the observer can feel if he is rotating and just say he restricts to cases where he is not rotating in order to carry out his measurements. Continuing then, for a non-rotating observer, the separation or tidal acceleration field in a geometric theory of gravity is the essence of the gravitational field. That is, if an observer at event $m \in M$ has velocity u , with $g(u, u) = -1$, then according to (3.3) and (3.4) we should interpret $R(u, u)$ as the negative flat Euclidean space divergence of the infinitesimal gravitational tidal acceleration field as seen by that observer at $m \in M$ who thinks his space is flat Euclidean. In any case, we shall henceforth simply refer to these facts as meaning that $Ric(u, u)$ is the Euclidean (space) negative divergence of the observer's infinitesimal tidal acceleration field when his velocity is $u \in T_m M$. Next, we consider how this relates to Newton's Law of gravity.

4. NEWTON'S LAW OF INFINITESIMAL TIDAL ACCELERATION

Let us briefly review how Newton's Law of gravity is formulated and how it can be recast in terms of the infinitesimal tidal acceleration field. Here we have a Euclidean space, E , of dimension n and a smooth time dependent gravitational vector field \mathbf{f} defined on an open subset U of E , where U is just E with possibly a finite set of points removed which represent point masses. Thus the energy density ρ is a smooth function on U and, in case $n = 3$, Newton's Law says that $div_E \mathbf{f} = -4\pi G\rho$ and $\mathbf{f} = \nabla\Phi$. Now this last condition is easily equivalent to $curl \mathbf{f} = 0$, since U is simply connected. On the other hand, keeping in mind that \mathbf{f} represents an acceleration field (or force per unit mass), if $s \in T_m E = E$ is a separation vector, then $D_s \mathbf{f}(m)$ is the infinitesimal gravitational tidal acceleration with separation vector s as in [31], pages 272-273 (see also [39], pages 38-42). This means that the infinitesimal gravitational tidal acceleration field is just $A_m^{(grv)} = T_m \mathbf{f}$ viewed as a vector field on E . From (2.1) concerning (2.12) and the $curl$, we see that Newton's Law for gravity in terms of the infinitesimal gravitational tidal acceleration field $A_m^{(grv)}$ at m says simply $trace(A_m^{(grv)}) = -k_n \rho(m)$ and $A_m^{(grv)}$ is self-adjoint. Of course, this also makes sense for any spatial dimension n , as in (2.1). Here, k_n is a constant which only depends on the spatial dimension n . Since $A_u^{(geo)}$ is self-adjoint (2.7), the obvious way to geometrize gravity is simply to identify $A^{(geo)}$ with $A^{(grv)}$.

Thus on any Lorentz manifold M , we say that the Newton-Einstein Law for infinitesimal tidal acceleration holds at m for the observer $u \in T_m M$ provided that $trace(A_u^{(geo)}) = -k_n \rho_u(m)$, where we now view $A_u^{(geo)}$ as the observer's infinitesimal gravitational tidal acceleration and where $\rho_u(m)$ is the energy density of matter and fields other than gravity u observes at the event m . Now, one relativistic problem with Newton's Law for gravity is the fact that it amounts to instantaneous action at a distance which conflicts with relativity. We will assume that this problem is surmounted by *only* requiring the equation to hold at the observer's event m . He can say nothing about events other than his location event as far as the law of gravity is concerned. This means henceforth, by definition, and in view of (3.4) that the Newton-Einstein Law of infinitesimal tidal acceleration at $m \in M$ for the time-like unit vector $u \in T_m M$ simply states

$$(4.1) \quad Ric(u, u) = trace(\mathcal{K}(u, u)) = -trace(A_u^{(geo)}) = k_n \rho_u(m),$$

which for short we refer to simply as NEIL at m for observer $u \in T_m M$.

We say that NEIL holds at $m \in M$ provided that it holds for each observer $u \in T_m M$. In this way, we overcome the relativists claim that there should be no preferred observer. We say that NEIL holds on M provided that NEIL holds at each point of M . In this way we make the NEIL into a relativistic universal law

of gravity, which we think of as the Newton-Einstein Law of gravity. Notice that each observer only claims (4.1) to hold at his own event.

We have in fact almost arrived at the correct law, but relativists could claim we have failed to include all the source energy on the right hand side of the equation. That is to say, NEIL should be *corrected by requiring that the gravitational field energy density observed by each observer is also included in the source term energy density*. We could call this the corrected NEIL or the CNEIL, but instead we shall call it the Einstein-Hilbert-Newton Law of infinitesimal tidal acceleration or EHNIL. Anticipating the ability of observer u at m to find the energy density of the gravitational field, $\rho_u^{(grv)}(m)$, we say that the EHNIL holds at m for observer u provided that

$$(4.2) \quad Ric(u, u) = k_n[\rho_u(m) + \rho_u^{(grv)}(m)].$$

Naturally, we then say the EHNIL holds at $m \in M$ provided it holds for each observer $u \in T_m M$, and say that the EHNIL holds for M provided that it holds at each event $m \in M$. This then is our *universal law of gravitation*, the Einstein-Hilbert-Newton Infinitesimal Law of Gravity. Of course, this naturally leads to the question as to the energy density of the gravitational field which an observer sees at his location event, which we turn to next.

5. THE ENERGY DENSITY OF THE GRAVITATIONAL FIELD

In order to deal with the energy density of the gravitational field, we must first think in terms of the basic assumption of the geometric notion of gravity which is that "free test" particles must follow geodesics. To partially paraphrase J. A. Wheeler, spacetime tells matter how to move. That is its job. But spacetime is the physical manifestation of the gravitational field, so it is really the job of the gravitational field to tell matter how to move. Thinking anthropomorphically, if this is the case, then from the point of view of the gravitational field itself, it is happiest when all particles are following geodesics. In fact, we can imagine that in a limiting sense, if "all particles" follow geodesics, then the gravitational field is completely relaxed and contains no energy. It is only when we try to push a particle off of its geodesic that we feel the reaction of the gravitational field, and notice we feel it right at the location of the event of trying to push the particle off of its geodesic, thinking in the case where $n = 3$.

Thus, relativistically, we should think of the manifestation of tension in the gravitational field is particles not following geodesics. Now, if a particle is not following a geodesic, then it is because it is being acted on by a force which is not part of the gravitational field itself. Because by definition, gravity acts only through causing particles to follow geodesics, in the absence of "outside" forces. When a force acts to move a particle a certain amount off of its geodesic path, the force required to do so is proportional to the particles inertial mass, by definition, but in essence, this says the gripping energy of the gravitational field at the point where the particle is located is somehow related to the inertial mass of the particle. Accepting this, the density of this tension energy in the gravitational field should be related to the force density as manifested in pressures in various directions.

That is, the energy momentum stress tensor tells us the pressures as seen by any observer in various directions, so from the energy momentum stress tensor itself, we should be able to find the energy density of the gravitational field. For instance, the pressure you feel on your bottom when sitting in a chair is a manifestation of the energy density of the gravitational field at those points on your chair. In a sense then, we could say that if the surface of your chair were replaced by an infinitesimally thin slab sitting on top of an infinitesimally lower chair, then the mass energy of the slab required to hold you in place divided by the volume of the slab is a reflection of the energy density of the gravitational field there. What is the minimum mass which can take care of this job?

In fact, the material the chair is made of in some sense is a reflection of the energy density of the gravitational field right where your chair is located. Even primitive people have an intuitive idea of the strength of material needed to make a chair, and thus have a working idea of the energy density of the gravitational field. We should therefore think of the least mass energy of material required to make a chair as a rough measure of the energy density of the gravitational field where the chair is to be used.

More generally, imagine an observer located at $m \in M$, ghost-like inside a medium with energy momentum stress tensor T and suppose that his velocity at m is u . Then exponentiating $u^\perp \subset T_m M$, the orthogonal complement of u in $T_m M$, the pressures are given by the restriction of T to $u^\perp \times u^\perp$. This is a symmetric tensor on a Euclidian space so can be diagonalized, as pointed out in the mathematical preliminaries. Notice that this does not mean T itself is diagonalizable. Thus, there is an orthonormal frame (e_1, e_2, \dots, e_n) for u^\perp with the property that $T(e_a, e_b) = p_a \delta_b^a$, for $a, b \in \{1, 2, \dots, n\}$. It is customary to refer to these observed spatial eigenvalues of T as the principal pressures observed, and their average is referred to as the observed *isotropic pressure*, p_u . Thus, np_u is the sum of the principal pressures as seen by the observer with velocity u . Imagine scooping out a tiny infinitesimal box in M at m whose edges are parallel to these u -spatial principal axes of this spatial part of the energy momentum stress tensor. We can imagine putting infinitesimally thin $(n - 1)$ -dimensional reflecting mirrors for walls of the box and filling the box with laser beams reflecting back and forth in directions parallel to the edges of the box with enough light pressure in each direction to balance the force from outside on these reflecting walls.

In a sense, we have standardized a system to balance the pressures acting to disturb the gravitational field, so we *define* the energy density of the gravitational field as seen by our observer to be the energy density of the light in this little box. The fact that a photon has zero rest mass should mean that the light energy constitutes a minimum amount of energy to accomplish this task of balancing the gravitational energy. However, it is an elementary problem in physics to see that the energy density of the light along a given axis is exactly the pressure in that principal direction.

Let us review this simple argument, in case $n = 3$. Assume the coordinates are (t, x, y, z) for simplicity and the box edges are parallel to these axes with lengths $\delta x, \delta y, \delta z$, respectively. Assume that the laser beams parallel to say the x -axis contain N_x photons, each having spatial momentum P_x . In time δt , the photons travel a distance of $c\delta t$ and hence each such photon makes $(c\delta t)/(\delta x)$ reflections for a change in momentum of $2P_x$ for each reflection.

Thus the total momentum transfer to the two end walls perpendicular to the x -axis for the laser beams paralleling the x -axis is

$$(5.1) \quad \frac{2P_x N_x c \delta t}{\delta x}.$$

This means that the force exerted on the two end walls is $(2P_x N_x c)/(\delta x)$. But the total area of the two end walls is $2\delta y \delta z$, so the pressure on the end walls is

$$(5.2) \quad p_x = \frac{N_x P_x c}{V},$$

where $V = \delta x \delta y \delta z$ is the volume of the box. But the relativistic energy of a photon with momentum P_x is $P_x c$. Therefore, the total energy density due to the x -axis beams is exactly the pressure in the x -direction on the walls perpendicular to the x -axis.

If p_x is negative, a similar argument using opposite charge distributions on the opposite walls of the box along the x -direction would have the opposite walls behaving like a capacitor and again, elementary calculations (the freshman physics "pillbox" argument using Gauss' Law for electric flux) easily lead to the conclusion that the energy density of the electric field of the capacitor has the same absolute value as the negative stretching pressure of the medium, and here it seems that the energy due to this stretching pressure (like the pressure in a stretched rubber band) should count as negative energy. Thus, when pressure is negative, the pressure is serving to reduce the energy of the gravitational field as it is "working with" the gravitational field. That is, as we scoop out the matter to create the little box, if the pressure is negative in the x -direction, we scoop so as to leave opposite charge distributions on the opposite faces in such a way that the attraction of the opposite faces balances the negative pressure of the medium. We could imagine for instance in case the capacitor is overcharged, that allowing this scooped out capacitor to "snap shut" then supplies the capacitor energy to the gravitational field and also lowers the energy of the gravitational field. So in this case of negative pressure, it must be that the energy density should be negative just as is the pressure.

We can therefore take it to be the case that the principal pressure in the x -direction gives the energy density contribution for that direction in any case. Likewise for the other two axes, consequently we see that the total energy density in the box is the sum of the pressures that the beams and fields are balancing, that is the trace of the observer's spatial part of the energy-stress tensor, $p_x + p_y + p_z$.

More generally, in light of the preceding heuristic arguments, for any n , we *define* the gravitational energy density seen by the observer $u \in T_m M$ to be the sum of the principal pressures, $\rho_u^{(grv)}(m) = np_u$. We now state this as a formal postulate.

Postulate 5.1. GRAVITATIONAL ENERGY DENSITY POSTULATE. *At each event $m \in M$, each observer $u \in T_m M$ observes the energy density of the gravitational field as being $\rho_u^{(grv)}(m) = np_u$, the sum of the principal pressures of the source matter and fields other than gravity at event m .*

At this point we can notice that we are already dealing with a physically intuitive description of the gravitational field which implies the Cooperstock hypothesis which says the gravitational field has no energy in the vacuum, because indeed, in the vacuum there is certainly no pressure. That is obviously postulate 5.1 implies the Cooperstock hypothesis.

Finally here, we should mention that our heuristic argument involving the laser light box could be replaced by a similar argument where photons are replaced by any particle which travels at the speed of light as it then has zero rest mass and therefore obeys the same energy momentum relation $E = Pc$ as photons do. For instance, if we think of the energy of the gravitational field as residing in particles called gravitons which travel at the speed of light, and if we assume that gravitons are trying to maintain geodesic motion of all other matter particles via pressure, then the same result seems to hold. Thus, maybe np_u is the energy density of gravitons as seen by the observer u , and thus gravitons would have no energy in the vacuum, or more precisely, the vacuum contains no gravitons. Thus, maybe gravitons are the ultimate constituent particles of matter and fields.

6. THE ENERGY MOMENTUM STRESS TENSOR OF THE GRAVITATIONAL FIELD

In view of the results of the preceding section we can now prove our theorem on the energy density of the gravitational field.

Theorem 6.1. *If M is a Lorentz manifold and the covariant symmetric tensor T on M models the energy momentum stress tensor on M due to all matter and fields other than gravity, then assuming the Gravitational Energy Density Postulate 5.1 is equivalent to assuming the covariant symmetric tensor*

$$(6.1) \quad T_g = T - c(T)g$$

is the unique symmetric tensor giving the energy momentum stress tensor of the gravitational field.

For the proof, suppose that $m \in M$ is any event and $u \in T_m M$ is an observer at $m \in M$. Suppose that T is the second rank covariant energy momentum stress tensor for the matter and fields other than gravity. Our task is to find the covariant second rank symmetric tensor T_g , which gives the energy momentum stress of the gravitational field from our previous physical argument that every observer should see it as the sum of the principal pressures. Thus, by the observer principle and the gravitational energy density postulate 5.1, T_g is uniquely determined by the requirement that $T_g(u, u) = np_u$ for each time-like unit vector $u \in T_m M$ no matter which $m \in M$.

Now, applying (2.3) to compute $c(T)$ we find that

$$c(T) = -T(u, u) + np_u,$$

and therefore,

$$np_u = T(u, u) + c(T) = T(u, u) - c(T)g(u, u),$$

which is to say finally that the second rank symmetric covariant tensor

$$T_g = T - c(T)g$$

does indeed do the job.

We point out here, that in general, such uniqueness does not imply existence, but here we have existence of the required tensor we seek from equation (6.1) itself. That is really the assumption that there is a covariant symmetric tensor T giving the energy momentum stress tensor of all matter and fields other than gravity is also giving the existence of the tensor T_g through equation (6.1).

In the reverse direction, by (2.3), if we assume that (6.1) is the energy momentum stress tensor for the gravitational field, then the gravitational energy density postulate 5.1 is an immediate consequence. This completes the proof of the theorem 6.1.

In view of (6.1) we define the *total energy momentum stress tensor* of all matter and fields including gravity to be

$$(6.2) \quad H = T + T_g = 2T - c(T)g = 2[T - (1/2)c(T)g].$$

Thus, by Theorem 6.1 and the observer principle, we know H must be the symmetric tensor which should serve as the source term for the gravitation equation, since for every $m \in M$ and observer $u \in T_m M$ we have

$$(6.3) \quad H(u, u) = \rho_u(m) + \rho_u^{(grv)}(m)$$

7. THE DERIVATION AND PROOF OF THE EINSTEIN EQUATION

We are now in a position to state and prove our theorem as regards the Einstein equation.

Theorem 7.1. *If M is a Lorentz manifold and T is any covariant symmetric tensor field on M which models the energy momentum stress tensor of all matter and fields other than gravity, then the EHNIL(4.2) together with the Gravitational Energy Density Postulate 5.1 is equivalent to the assumption that the equations*

$$(7.1) \quad Ric = k_n H = k_n [2T - c(T)g] = 2k_n [T - (1/2)c(T)g]$$

hold on M with $H = (1/k_n)Ric$ being the total energy momentum stress tensor of gravity and all matter and fields. In particular, if $n = 3$ so spacetime is four dimensional, then automatically $div(T) = 0$ as a consequence of these assumptions. If n is not 3, then these assumptions and the assumption that $div(T) = 0$ imply that $dR = 0$, where R is the scalar curvature of M .

To prove the theorem 7.1 use theorem 6.1. Assuming the EHNIL(4.2) holds for all observers everywhere, we now have by (6.3),

$$(7.2) \quad Ric(u, u) = k_n H(u, u),$$

for every observer $u \in T_m M$ at $m \in M$, where k_n is a universal constant depending only on n . As an aside, beyond 6.1 and the EHNIL, the real reason behind everything here is the fact that $Ric(u, u)$ is the negative Euclidean divergence of the tidal acceleration (3.4), so in physical terms we are using the tracial identification of the gravitational tidal acceleration with the geometric tidal acceleration. But let us return to the proof. Thus, (7.2) merely says that any observer u at any $m \in M$ sees the EHNIL to hold. Since (7.2) is true for u being any time-like unit vector, by the observer principle (corollary 2.1), we must have (7.1) as an immediate consequence.

For the reverse direction, if we assume that the equation (7.1) holds with H giving the total energy momentum stress tensor of all gravity all matter and all fields, then as T is the energy momentum stress tensor of all matter and fields other than gravity, and as $H = 2T - c(T)g$, we must have

$$T_g = H - T = T - c(T)g$$

which by Theorem 6.1 then implies the gravitational energy density postulate 5.1 and then (7.2) holds for any observer u which now by (6.3) says the EHNIL(4.2) holds for all observers.

In case $n + 1 = 4$, it is customary to write $k_3 = 4\pi G$, so then

$$(7.3) \quad Ric = 4\pi GH = 8\pi G[T - (1/2)c(T)g], \quad n = 3,$$

which is a well-known form of Einstein's equation. As $c(g) = n + 1$ and $c(Ric) = R$, where as usual, R is the scalar curvature, we find that $R = (1 - n)k_n[c(T)]$, so the equation can be also written as $Ric = k_n[2T] + (1/(n - 1))Rg$, and this results in

$$(7.4) \quad Ric - \frac{1}{n - 1}Rg = 2k_n T.$$

The energy density tensor $T_g = T - c(T)g$ can be expressed in terms of the Ricci tensor and scalar curvature using (7.4) and the result is

$$(7.5) \quad T_g = \left(\frac{1}{2k_n}\right)[Ric + \left(\frac{1}{n - 1}\right)Rg].$$

As usual, we define the *Einstein tensor* by

$$Einstein = Ric - (1/2)Rg,$$

which has the property that

$$div(Einstein) = 0,$$

no matter the value of n . But in case $n = 3$, we find that the left hand side of (7.4) is the Einstein tensor. In this case, with $k_3 = 4\pi G$, we find the most familiar form of the Einstein equation

$$(7.6) \quad Einstein = Ric - (1/2)Rg = 8\pi G T, \quad n = 3.$$

Notice that we have not used local conservation of energy, $div(T) = 0$. Since the left side of (7.6), the Einstein tensor, *Einstein*, is divergence free, we find $div(T) = 0$ as a consequence of our derivation, in the case where $n = 3$. On the other hand, it appears that for n not equal to 3 we would have that $div(T)$ is in general not zero. That is, it is only in spacetime dimension 4 that the energy momentum stress tensor of matter and fields other than gravity can be infinitesimally conserved without automatically putting severe restrictions on spacetime. Specifically, in case $n + 1$ is not equal to 4, we find immediately that $div(T) = 0$ implies $dR = 0$, and hence the scalar curvature of spacetime must be constant if the energy stress tensor of matter and fields other than gravity has zero divergence.

To see this, in more detail, a simple calculation shows that for any smooth scalar function f we have $div(fg) = df$, since $\nabla g = 0$. The Einstein tensor has vanishing divergence in any dimension (due to the second Bianchi identity), and this is clearly equivalent to $div(Ric) = (1/2)dR$. Therefore, taking the divergence of both sides of (7.4) gives

$$\left(\frac{1}{2} - \frac{1}{n - 1}\right)dR = 2k_n div(T).$$

Thus, if we assume that $div(T) = 0$, then either $dR = 0$ or $n = 3$, and on the other hand, if we assume $n = 3$, then as the Einstein tensor has vanishing divergence, then so must T . This completes the proof of our theorem 7.1.

Before proceeding further, we should remark that it is often thought that as the Weyl curvature need not vanish in the vacuum, that it should enter into the expression for T_g . However, we have a physical expression for the energy density of the field as seen by any observer, so that determines what the expression for T_g will be. If the Weyl curvature is not part of the result, we must accept the fact that Weyl curvature cannot generate gravitational energy, on this view.

Here, with (7.5), we see explicitly that the gravitational field energy momentum stress tensor does not depend on the Weyl curvature. Rather, it only depends on the Ricci tensor. Thus, our Theorems 6.1 and 7.1, together with our gravitational energy density postulate 5.1 in particular guarantees that the Weyl curvature does not generate gravity. The Einstein equation determines the Ricci tensor directly from the matter energy momentum stress tensor.

However, we should keep in mind that including appropriate boundary conditions, when $n = 3$, the Einstein equation determines the full Riemann curvature tensor, therefore including the Weyl curvature tensor. In particular, the reader should note equations (4.28) and (4.29) on page 85 of [22] for the Weyl curvature tensor, which follow from the Bianchi identities and are similar in form to the Maxwell equations for the electromagnetic field tensor. Thus, the Weyl curvature which gives the curvature in the vacuum, as the Ricci curvature vanishes in the vacuum, is contained in the boundary conditions under the Einstein equation.

We will until further notice now restrict to the case $n+1 = 4$, so we have ordinary spacetime, and therefore $\text{div}(T) = 0$.

In the case of $n+1 = 4$, if the condition that $\text{div}(T) = 0$ is dropped in the usual derivation where one equates T to a linear combination of the metric tensor, the Ricci tensor and the product of the scalar curvature with the metric tensor, and only requires the time components give Newtonian gravity in the Newtonian limit, the result is a generalization of the Einstein equation with a new free parameter which when equal to 1 gives the usual Einstein equation. This has been investigated as to its ramifications for cosmology [1]. But, this does mean that the assumption $\text{div}(T) = 0$ is necessary for this type of derivation of the Einstein equation, since without it the free parameter may be other than unity.

From (7.5) we now have

$$(7.7) \quad T_g = (1/8\pi G)[\text{Ric} + (1/2)Rg] = (1/8\pi G)(\text{Einstein} + Rg).$$

From the last expression on the right, we see, as T and the Einstein tensor, $\text{Einstein} = \text{Ric} - (1/2)Rg$ both have zero divergence, that

$$(7.8) \quad \text{div}(H) = \text{div}(T_g) = (1/8\pi G)\text{div}(Rg) = (1/8\pi G)dR = -d[c(T)].$$

So even though the total energy and gravitational energy are not infinitesimally conserved, the divergence is simply proportional to the exterior derivative of the scalar curvature. Of course, $\text{div}(T_g) = -d[c(T)]$ is obvious from the definition, (6.1), once we accept $\text{div}(T) = 0$. In particular, as $d^2 = 0$, this means that

$$(7.9) \quad d[\text{div}(H)] = d[\text{div}(T_g)] = 0,$$

but (7.8) is even better as it shows $\text{div}(T_g)$ is an exact 1-form on M .

On the other hand, the equation (7.8), when written

$$(7.10) \quad \text{div}(T_g) + d[c(T)] = 0$$

has another interpretation. In classical continuum mechanics written in four dimensional form of space plus time, the divergence of the energy stress tensor equals the density of external forces. Of course in relativity, the energy momentum stress tensor T contains everything and there are no external forces, as gravity is not a force. But, we can view (7.10) as saying that from the point of view of the gravitational field, the matter and fields represented by T are acting on the gravitational field as an external force density of $-d[c(T)]$. In classical continuum mechanics, the external force density has zero time component, but relativistically such is not the case, the force only has zero time component in the instantaneous rest frame of the object acted on. We can therefore view (7.10) as saying that the divergence of the the gravitational field's energy stress tensor is being balanced by the rate of increase of $-c(T)$. If p_x, p_y, p_z are the principal pressures in the frame of an observer with velocity u , where $g(u, u) = -1$, then $\rho_u = T(u, u)$ is the energy density observed, and $\text{div}(T_g)(u)$ is then the power loss density of the gravitational field.

Now $c(T) = -\rho_u + p_x + p_y + p_z = -\rho_u + 3p_u$, where p_u is the isotropic pressure, so (7.10) becomes

$$(7.11) \quad \text{div}(T_g)(u) = D_u \rho - 3D_u p_u.$$

Thus, the observer sees the divergence of energy of the gravitational field is exactly the rate of increase of energy density of the matter and fields less the rate of increase of principal pressures. In particular, in any dust model of the universe (pressure zero), the gravitational energy dissipation is exactly balanced by the rate of increase of energy density of the matter and fields. If T is purely the electromagnetic stress tensor in a region where there are only electromagnetic fields, then $c(T) = 0$, and the gravitational energy-stress tensor has zero divergence, so is then infinitesimally conserved.

To compare our energy momentum stress tensor of the gravitational field with the various gravitational energy pseudo-tensors, keep in mind that all examples of such gravitational energy pseudo-tensors can be made to vanish by appropriate choice of coordinates and therefore cannot represent real energy of any kind in relativity. Such pseudo-tensors generally obey coordinate conservation laws in appropriately chosen coordinates making them useful in certain calculations, but they cannot represent real energy as in relativity, real energy cannot just be transformed away by some choice of coordinates. By contrast, our T_g is fully covariant and represents localizable gravitating energy density as seen by each observer, but in general is not conserved, as $div(T_g)$ may not vanish in general. The exact calculation of the difference between the various pseudo-tensors and T_g in various examples should be an interesting problem for future research.

Looking back at the derivation, one can now see that if there is a distinction between active gravitational mass and inertial gravitational mass, then in equation (4.2), the first of the two terms is the energy density due to active gravitational mass and the second of the two terms being the sum of principal pressures is therefore an inertial mass, as it is inertial mass not following geodesic motion which creates pressure. This would mean that in the equation $Ric = 4\pi G[T + T_g]$ the second term on the right is the tensor which has the inertial mass whereas the first term is the term with the active gravitational mass. But, this would seem to lead to a violation of the principle of relativity, as the pressures would be different in different reference frames leading to conversion between active gravitational and inertial masses depending on the observer. Thus, this derivation seems to indicate the equality of active gravitational mass with inertial mass, a point which is not addressed in the usual derivations of Einstein's equation. Possibly an improved version of this derivation might derive the equality of inertial and active gravitational mass. Of course, the geodesic hypothesis itself makes the passive gravitational mass equal to the inertial mass, which seems to be the reason why the problem of equality of active gravitational and inertial mass is often overlooked in elementary treatments of general relativity.

Finally here, we should point out that Einstein's original Equivalence Principle is often misconstrued to say that gravitational fields can be transformed away by choice of coordinates, and this is certainly not the case, as Frank Tipler has stated on many occasions. This is well known to experts in general relativity. Gravity in general relativity is curvature of spacetime, and curvature cannot be transformed away. If we view connection coefficients as "gravitational forces", then using normal coordinates at a point makes them disappear, but this merely reflects the fact that gravitational forces do not exist in general relativity, virtually by definition. Einstein used the example of an accelerating coordinate system to effectively transform away a uniform gravitational field in which there is no actual curvature of spacetime and therefore no real gravity. At each event, given a specified limit in level of measurement accuracy, there is a neighborhood in which curvature effects cannot then be measured, and in such neighborhoods of an event, the equivalence principle may be used effectively. One must be careful of subtle pitfalls. For instance, when Einstein used the elevator thought experiment to reason that light would bend in a gravitational field, he was using the fact that in the accelerated reference frame the null geodesics appear curved and then generalizing to arbitrary gravitational fields.

In fact, the elevator thought experiment merely gives the result for the bending of light that Newtonian gravity in flat Euclidean space would give under the assumption that photons have inertial mass. It takes the full Schwarzschild solution to arrive at the correct answer for the bending of light, which Einstein fortunately realized before the experimental measurements were made.

Another way to look at this light bending problem would be that in NEIL we have out in the near vacuum of space that for the light beam the law of gravity is $Ric = 4\pi G T_{EM}$ where T_{EM} denotes the energy momentum stress tensor of the electromagnetic field. But, as $c(T_{EM}) = 0$, we have $(T_g)_{EM} = T_{EM}$, so Einstein's equation, EHNIL, becomes $Ric = 4\pi G [T_{EM} + T_{EM}] = 4\pi G [2T_{EM}]$ which means that the

photon's electromagnetic field gives twice the curvature of the spacetime at points along its track as would be the case in NEIL, which should reasonably lead to the doubling of the bending angle. Of course this is a nonsense argument, since the light bending has to do with tracks of null geodesics in the gravitational field of a large gravitating object and not the gravitational field of an electromagnetic wave itself. But, if we think of a photon passing a planet, theoretically, we are allowed to think of the planet as following a path in the gravitational field of the photon, and the preceding analysis says the planet's path should be bent twice as much in Einstein's theory as in Newton's theory, so reciprocally, the photon's track should be bent twice as much. Maybe the argument is not so specious, and should be examined further. On the other hand, this does tell us that the effective gravitational mass of pure electromagnetic radiation or laser light is double its inertial mass, which possibly could be detected using powerful lasers in an inertial confinement fusion laboratory, thus leading to another test of Einstein's theory. Tolman ([46], Chapter VIII) has noticed the prevalence of this doubling effect for electromagnetic radiation in many examples, all calculated using the weak field approximation. But, now using Theorems 6.1 and 7.1, we see that the EHNIL is telling us the effective gravitational mass-energy of electromagnetic fields is very generally double the inertial mass-energy.

More generally, our conclusion here is that $\rho_u + 3p_u$ is the effective gravitational mass-energy density observed by an observer with velocity u . For that is what is dictated by Einstein's equation, since it is equivalent to the EHNIL and the gravitational energy density postulate, by Theorem 7.1.

8. THE GRAVITATION CONSTANT G

So far, we have not said anything about the determination of the gravitation constant G . To evaluate this, we merely need to check the results of experiments with attractive "forces" between masses. But it is much simpler to just use Newtonian gravity in an easy example where the results should be obviously approximately the same. Consider an observer situated at the center of a spherical dust cloud of uniform density ρ , and calculate the tidal or separation acceleration field using Newton's law of gravitation. We can observe here that the energy momentum stress tensor satisfies $T(v, w) = \rho g(v, u)g(v, w)$ where u is the velocity field of the dust cloud. Thus we calculate easily that $T_g(u, u) = 0$ meaning that a co-moving observer sees the gravitational field as having energy density zero. In this case, the NEIL and EHNIL coincide for u and thus as it seems reasonable that the NEIL should have the Newton gravitation constant as its constant, then that means $G = G_N$.

It is easy to give a more elementary argument here. At distance r from the center, but inside the cloud, the mass acting on test particles at radial distance r is simply the mass inside that radius, $M(r)$, by spherical symmetry, as is well-known in Newtonian gravitation. Here, we have $M(r) = (4/3)\pi r^3 \rho$.

But Newton's Law says the acceleration of a test mass near the center of the dust cloud is radially inward, and if r is the distance from the center, then the radial component of acceleration is given by

$$(8.1) \quad a_r(r) = -G_N \frac{M(r)}{r^2} = -G_N \frac{4\pi\rho r}{3}.$$

Here, G_N is the Newtonian gravitation constant.

On the other hand, considering an angular separation of θ , the spatial separation is $s = r\theta$, so the relative acceleration of nearby test particles in the s - direction perpendicular to the radial direction is therefore

$$(8.2) \quad a_s(r) = \theta a_r(r) = -G_N \frac{4\pi\rho r\theta}{3} = -G_N \frac{4\pi\rho s}{3}.$$

Thus the rate of change of separation acceleration of nearby radially separated test particles in the radial direction at given r is by (8.2),

$$(8.3) \quad \frac{da_r}{dr} = -G_N \frac{4\pi\rho}{3},$$

whereas in the s direction we have the rate of change of separation acceleration is

$$(8.4) \quad \frac{da_s}{ds} = -G_N \frac{4\pi\rho}{3},$$

the same result again. But there are two orthogonal directions perpendicular to the radial, so now we see that if \mathbf{a}_u denotes the spatial or tidal acceleration field around our observer at the center of the dust cloud, then

$$(8.5) \quad \text{div}_u(\mathbf{a}_u) = -G_N 4\pi\rho.$$

As we are dealing with dust, the pressures are zero, so there is no gravitational energy density, and thus ρ is now the total energy density seen by our observer. Thus, we have by (3.4), that $\text{Ric}(u, u) = G_N 4\pi\rho = 4\pi G_N H(u, u)$. But now comparing this result with (7.2), with $k_3 = 4\pi G$, we see that we must have $G = G_N$.

Notice that in our development, we have used the observer principle as a form of the principle of general relativity to reduce everything to working with the time component in an arbitrary frame for the tangent space. The trick is to be able to work completely generally so that conclusions apply to T_{00} and Ric_{00} no matter the frame, even in a non-coordinate frame, which seems best expressed by using $T(u, u)$ and $\text{Ric}(u, u)$, to remind us that we are dealing with an arbitrary time-like unit vector. It is only now at the end once we have Einstein's equation that we allow a calculation in a special frame in order to evaluate the gravitation constant.

Consider for a moment the derivation of Einstein's equation given in [19]. In effect, the derivation of the Einstein equation given in [19] uses the analysis (adapted from arguments of Tolman [46]) of the special case of a static arrangement of mass for a gravitating fluid drop and adds the Newtonian gravitational energy density of the fluid drop as expressed in terms of pressure through the requirement that its surface pressure be zero to get the time component of the Einstein equation. Since the setup is a special arrangement of mass, one cannot assert the observer principle, because the only observer for which the equation works is the special observer moving with the drop. However, one can appeal to the general covariance desire of relativity that equations should be tensor equations valid in all frames, from which one surmises that if you have found an equation relating the time components in a special frame, then the other components in that special frame should also be equal. Once you accept the full tensor equation in any frame, then it is valid in all frames and you next surmise that if it works for the liquid drop, then it must work in general. But, in our present situation, we have the full equation, in complete generality, and can simply go backwards through the development in [19] to see that the time component of the equation in the liquid drop case is Newton's law, and therefore again conclude that our G in (7.6) is identical to the Newtonian gravitational constant. For a treatment of linearized Einstein gravity and its Newtonian approximation in general, one can consult [31] or [49].

At this point, we can simply choose units such that $G = 1$ and we henceforth drop this factor from the equation for simplicity.

9. THE EINSTEIN DERIVATION

It is interesting that Einstein realized fairly early in his search for the gravitation equation that the vacuum equation should be $\text{Ric} = 0$. This led him to try the equation $\text{Ric} = 4\pi G T$, as the general gravitation equation when matter is present. In fact, this equation obviously results from the observer principle if we assume spacetime satisfies NEIL instead of EHNIL, that is if our observers neglect the energy density of the gravitational field. He soon rejected this as not being compatible with reality, partly due to the fact that it would require that $\text{div Ric} = 0$.

He also knew that the energy density of the gravitational field should be included in the source, so if he had found the expression we have for the energy stress tensor of the gravitational field, he would have surely arrived at the final equation at this time.

As it was, in summary, he finally [17] took the already accepted vacuum equation $\text{Ric} = 0$ and for special coordinate frames, he was able to rewrite the vacuum equation in the form $s = k(t - (1/2)c(t)g)$ where t is his pseudo-tensor whose coordinate divergence is zero, and where s itself is a coordinate divergence of a

third rank pseudo-tensor. Let us call a coordinate system *isotropic* provided that in these coordinates we have $\det(g_{\alpha\beta}) = -1$. Thus Einstein found it useful to restrict to isotropic coordinates.

In fact, he found $Ric = s - k(t - (1/2)c(t)g)$, to be true in any isotropic coordinate system. He therefore interpreted t as the energy density of the vacuum gravitational field and interpreted the new form of the vacuum equation as making t the source. He then merely guesses that in the presence of matter with energy momentum stress tensor T the source should be $t + T$ instead of merely t . Thus, when t is replaced by $t + T$ in the new form of the vacuum equation we have $s = k[(T + t) - (1/2)c(T + t)g]$ as the candidate for the general non-vacuum equation. We then see that moving the terms involving the pseudo tensor back to the left side of the equation results in $Ric = k[T - (1/2)c(T)g]$, true in any isotropic coordinates. But this last equation is a fully a covariant equation.

In a sense, his derivation begins with and is based on the pseudo tensor for the energy density of the gravitational field. Technically his equation was $-Ric = k[T - (1/2)c(T)g]$, because he used a metric with signature $(+ - - -)$. Of course, our summary has left out the Hamiltonian method he used to arrive at his pseudo tensor and the considerable technical calculations required to arrive at the vacuum equation in terms of the pseudo tensor. But, in outline, it is really quite a nice derivation, and in many ways superior to most of the modern derivations.

The fact that the coordinate divergence of the pseudo tensor vanishes means that the general Stokes' theorem (sometimes in this particular setting called the divergence theorem or Gauss' theorem) can be applied to give macroscopic conservation of gravitational energy. On the other hand, once the source t is replaced by $t + T$, it is no longer the case that this latter gravitational pseudo tensor has vanishing coordinate divergence, so the gravitational pseudo tensor loses its conservation law in the presence of matter. It is rather Einstein's total energy stress pseudo tensor, $T + t$, material and gravitational, whose coordinate divergence vanishes. It seems this lead Einstein to question the need and even the validity for general covariance in his formulation, since the vanishing of the covariant divergence could not be integrated to give any macroscopic conservation law.

In our opinion, the real major weakness in the argument is the reliance on a variational argument using a Hamiltonian to obtain the pseudo tensor, since there is no apparent way to justify this, other than picking something that seems simple out of thin air. Specifically, he chose the integrand to be $g^{\mu\nu}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta}$ for his variational integral, where here we can take $\Gamma_{\beta\gamma}^{\alpha} = \omega^{\alpha}(\nabla_{e_{\gamma}}e_{\beta})$, with (e_{α}) the coordinate frame basis and with (ω^{α}) the corresponding dual frame basis.

For instance, in the Hilbert argument using the scalar curvature as the integrand, it is certainly simple to write down and after the fact, it does give the correct equation. But, what is the physical basis for choosing the scalar curvature for the variational argument? Without any physical justification, we have to admit it is just a lucky guess based on trying the simplest thing, which, of course, is always a good idea when you have nothing else to go on. Just because you try something simple and it happens to work does not mean you understand why it works. After nearly a century of general relativity, we are quite confident of the results of action principles in general relativity, but for deriving the equation, it is unsatisfactory. For instance, the Einstein derivation evolved out of Einstein's consideration of various physical problems and possibilities and he happened to arrive at the result at almost the same time as Hilbert. Now if Hilbert had proposed his action method two years earlier, would Einstein have believed the equation was the correct equation? Maybe and maybe not. It is putting the cart before the horse. In fact, setting $Einstein = E$, if we simply want a simple derivation, as $E(u, u)$ is half the scalar curvature of W_R , the exponential Riemannian space orthogonal to u , for any time-like unit vector u , the simplest derivation is just to guess each observer sees his spatial curvature proportional to his observed mass density with a universal constant of proportionality. This immediately gives $E(u, u) = 2k_3T(u, u)$, for each time-like unit vector u from which we conclude $E = 2k_3T$, for some constant k_3 , by the observer principle. Instant derivation of the Einstein equation. But why should we have spatial curvature proportional to energy density? If you are aware of the observer principle, it is the obvious guess, but you have no way to know you are correct, since there is no physics in the argument-it is just mathematics.

We are not claiming that the Einstein Hilbert Lagrangian method has no value. It surely has value for certain calculations, especially since the Lagrangian terms for many fields are known and can be added

in to the calculations. We are simply pointing out, that if you did not know the equation before such a derivation, you still might not be convinced. The derivation we have presented here seems to have the convincing property that it is the only way to very generally and naturally "push" Newton's Law of gravity into a general relativistic framework which includes the energy density of the gravitational field as seen by all observers. For instance, any mathematician familiar with Newton's Law of Gravity and basic differential geometry would be convinced by it if he accepts that each observer with velocity u should see $3p_u$ as the energy density of the gravitational field. It would seem to us that the fact that $\text{div}(T) = 0$ is an immediate consequence of this derivation makes it all the more attractive and convincing.

10. THE COSMOLOGICAL CONSTANT

If we include the cosmological constant Λ in the Einstein equation, it becomes

$$(10.1) \quad \text{Ric} - (1/2)Rg + \Lambda g = 8\pi T,$$

which is of course the same as

$$(10.2) \quad \text{Ric} - (1/2)Rg = 8\pi[T - (1/8\pi)\Lambda g],$$

which means we view the equation here as having a modified energy momentum stress tensor

$$(10.3) \quad T_\Lambda = T - (1/8\pi)\Lambda g.$$

We then have $c(T_\Lambda) = c(T) - (1/2\pi)\Lambda$, so the effective energy momentum stress tensor of the gravitational field is

$$(10.4) \quad (T_\Lambda)_g = T - c(T)g + (3/8\pi)\Lambda g = T_g + (3/8\pi)\Lambda g,$$

and the effective total energy momentum stress tensor serving as source is

$$(10.5) \quad H_\Lambda = 2T - c(T)g + (1/4\pi)\Lambda g = H + (1/4\pi)\Lambda g.$$

In any case, as $\text{div } g = 0$, it follows that our conclusions about the energy-momentum flow of the gravitational field from (7.10) and (7.11) remain valid, even in the presence of a cosmological constant. Equations (10.4) and (10.5) are corrections of equations (8.4) and (8.5) of [12] where the numerical coefficients of the Λg terms were incorrectly given as $1/2\pi$, in both cases.

11. QUASI LOCAL MASS

The problem of defining the energy contained in a space-like hyper-surface has led to many different definitions of the mass enclosed by a closed space-like surface contained in an arbitrary spacetime manifold, and these go by the general name quasi-local mass. Typically, they are defined by some kind of surface integral and give an indication of the mass enclosed by the space-like surface. One of the oldest is known as the Tolman integral and is advocated by Fred Cooperstock [7], [46] (see also [28], equation (100.19), as well as [32], [33]). For an extensive survey of these we refer the interested reader to [48]. In particular, the results of [44] on the Penrose quasi-local mass show that the results can be interesting when the space-like surface is not the boundary of a space-like hyper-surface.

A list of desirable properties of any definition of quasi-local mass is given in [30], where in particular it is shown that for their definition, the quasi-local mass enclosed by a space-like surface S is non-negative provided that the dominant energy condition holds and the surface S is the boundary of a hyper-surface, Ω . It is further assumed that the boundary surface S has positive Gauss curvature and space-like mean curvature vector, and consists of finitely many connected components. The local energy condition assumed (equivalent to the dominant energy condition) is framed in terms of the second fundamental form of the hyper-surface, and in particular, we can see that for a geodesic hyper-surface it reduces to the condition that

the scalar curvature of the hyper-surface, Ω , is non-negative, since in that case the second fundamental form vanishes (extrinsic curvature zero). But, in this case, the scalar curvature of the space-like hyper-surface Ω is $2Einstein(u, u) = 16\pi T(u, u)$, where u is a time-like future pointing unit normal field on Ω . So if the energy momentum stress tensor satisfies the weak energy condition in this case, then the energy density as seen by observers riding the hyper-surface is non-negative, and we would simply integrate $(1/8\pi)Einstein(u, u)$ over the hyper-surface to find the energy inside, which is clearly non-negative.

The amazing result in [30] is that the quasi-local mass defined there, which is defined in terms of integrals over the boundary S , is non-negative under the dominant energy condition. For instance, their results show if the energy inside any one component of S vanishes, then S is connected and Ω is flat ([30], Theorem 1, page 183), and thus the result shows that the energy in Ω is in some sense determined by the geometry of the boundary and its mean curvature vector under the assumptions stated above.

The small scale and large scale asymptotic properties are analyzed in [53], and in particular, in the vacuum the result is that to fifth order the quasi-local mass for small spheres is asymptotic to the Bel-Robinson tensor whereas in general to third order it is asymptotic to the energy momentum stress tensor of matter times volume. Unfortunately, there are drawbacks to this definition of quasi-local mass, as pointed out in [36], and it seems the situation is improved with the later treatments of Wang and Yau, in [50], [51], [52].

Let us use our total energy momentum stress tensor to formulate an invariant approach to quasi-local energy. If we have an open subset U of M and a time-like unit vector field u defined on U , we can think of the integral curves of u as being the histories of a field of observers. We can then form $H(u, u)$ as a function on U and assuming orientability of U we can choose a normalized volume form μ_U so that $\mu_U(u, e_1, e_2, e_3) = 1$, for (u, e_1, e_2, e_3) any local positively oriented orthonormal frame. The natural way to proceed here seems to be to form a type of action integral which we can call the *mass action integral*:

$$(11.1) \quad A(u, U) = \int_U H(u, u)\mu_U = \frac{1}{8\pi G} \int_U Ric(u, u)\mu_U.$$

The strong energy condition says $Ric(v, v) \geq 0$ for any time-like vector v , and thus if this condition is satisfied, then clearly the only way that the action integral can vanish is for $R(u, u)$ to vanish on U . But, this does not seem to obviously allow us to conclude that $Ric = 0$ on U . However, on physical grounds, it should allow us to conclude $Ric = 0$ on U . That is, if we fill spacetime with observers everywhere, then if nobody observes any gravitating energy, there should be none. So this becomes then a natural mathematical conjecture. In any case, it would seem that this mass action should be the invariant means for constructing quasi-local mass.

In order to make use of the total energy-stress tensor, H , in a setting similar to that of Liu and Yau, [30] or Wang and Yau [50],[51], one would assume an appropriate energy condition, and then for a space-like hyper-surface K with future time-like unit normal field u , it is natural to consider $H(u, u)\mu_K$ where μ_K is the volume form due to the Riemannian metric induced on K . The integral of $H(u, u)\mu_K$ over all of K should be the total energy inside K .

More generally, if we assume that H is dominantly non-negative, that is, it satisfies the analogue of the dominant energy condition for T , then given another reference future pointing time-like vector field k , one might then integrate $H(u, k)\mu_K$ over K . If a 2-form α can be found on K satisfying $d\alpha = H(u, k)\mu_K$, and if K is a 3-submanifold with boundary B , then by Stokes' theorem, the total energy inside K is related to the integral of α over the boundary B of K .

In particular, we say that K is *instantaneously static* if there is an open set $U \subset M$ containing K and a vector field k on U which is future pointing and orthogonal to K and which satisfies Killing's equation, at each point of K . If $\omega = k^*$ is the dual 1-form to k , so $\omega(v) = g(k, v)$ for all vectors v , then this is equivalent to requiring $Sym(\nabla\omega)|_K = 0$ or equivalently that $(d\omega)|_K = 2\nabla\omega|_K$, which to be perfectly clear means that the difference $d\omega - 2\nabla\omega$ as calculated on U in fact is zero at each point of K . Then as in the Komar [24] integral (see [40], [49], pages 287-289 or [41], pages 149-151) it follows that

$$(11.2) \quad (-1/8\pi)d * d\omega = (1/4\pi)Ric(u, k)\mu_K = H(u, k)\mu_K.$$

Here, $*$ denotes the Hodge star operator on M . Thus, $(-1/8\pi) * d\omega$ is a potential for the total energy on K . For any closed 2-submanifold S of K we define the quasi-local total energy $H(S, k)$ by

$$(11.3) \quad H(S, k) = -\frac{1}{8\pi} \int_S *d(k^*).$$

Thus, if $K_0 \subset K$ is a submanifold with boundary $S = \partial K_0$, then by Stokes' Theorem, (11.3) becomes

$$(11.4) \quad H(S, k) = -\frac{1}{8\pi} \int_{K_0} d * d(k^*) = \int_{K_0} H(u, k) \mu_K,$$

which is then non-negative if the strong energy condition holds. Thus, if $H(S, k) = 0$, with $S = \partial K_0$, then by (11.4), under the assumption that the strong energy condition holds, we would conclude that $Ric(u, k) = 0$ on K_0 . But, this means that $Ric(u, u) = 0$ on K_0 , which means that none of the observers in the field detect any energy.

Notice that if we have an asymptotically flat spacetime with a global time-like Killing vector field orthogonal to a spacelike slice, normalized to be a unit vector at spatial infinity, then our definition of the quasi-local total energy would be exactly the Komar mass which is well known in the literature [48]. Thus in the expression $H(S, k)$, the normalization for k is determined by requiring that it be of unit length at the event at which the observer is located. If the observer is located so that S is in the observer's causal past, then it would seem we must assume that the domain of k contains this past light cone.

In general, if k is a Killing field on all of the open set U , then being orthogonal to K means ([49], page 119, (6.1.1)) that also $\omega \wedge d\omega = 0$, where $\omega = k^*$. Then (see [49], page 443, (C.3.12)) we find, using $f = \ln(|g(k, k)|)$,

$$(11.5) \quad d\omega = -\omega \wedge df,$$

and using the fact that here $*[\omega \wedge df] = -(e^{f/2} D_n f) \mu_S$, where n is the outward unit normal to $S = \partial K_0$, and $\mu_S = dA$ is the area 2-form on S , we obtain finally,

$$(11.6) \quad H(S, k) = -\frac{1}{8\pi} \int_S e^{f/2} D_n f dA.$$

In particular, for the vacuum Schwarzschild solution with mass parameter \mathcal{M} , taking the Killing field $k = \partial_t$, we see easily that the mass calculated using the integral (11.6) gives the value \mathcal{M} for the mass enclosed by any sphere centered at the "origin" when we normalize the Killing field to be a unit vector at infinity. On the other hand, if we calculate that value of the integral by normalizing to make the Killing vector a unit at radial coordinate r_0 , as $H(S, k)$ is homogeneous in k , the normalizing constant comes out resulting in

$$(11.7) \quad \mathcal{M}_{r_0} = \frac{\mathcal{M}}{[1 - \frac{2\mathcal{M}}{r_0}]^{1/2}}.$$

Keeping in mind this is now the total energy, gravitational and massive, this indicates a problem develops as $r_0 \rightarrow 2\mathcal{M}$, even though we know it is not a real problem for the spacetime. The problem is probably due to the normalization involving the Schwarzschild radial coordinate which obviously breaks down at $r_0 = 2\mathcal{M}$. After all, what we are integrating is equivalent by Stokes' Theorem to integrating $H(u, k) \mu_K$, when $S = \partial K_0$, and we really want to be integrating $H(u, u) \mu_K$. We do not have the actual potential. On the other hand, this does seem to reflect correctly the fact that as one approaches the horizon of a black hole it takes infinite force to keep from falling in.

Let us now use these results to compute the mass action integral (11.1). To do this, let us assume that U is foliated by spacelike submanifolds determined by the Killing parameter t on U , so the leaves are the level manifolds of t , and that u is orthogonal to each leaf of this foliation. We assume that $k = hu$ is the Killing vector field on all of U , where h is the redshift factor. Assume now that K is a compact 4-submanifold of U

with boundary ∂K and that the intersection of K with the leaf at time t is K_t with boundary ∂K_t which is the intersection of ∂K with the leaf at time t , for $t_1 \leq t \leq t_2$. Then $k^* = hu^*$ and we see the volume form on K can be expressed as $\mu_K = \mu_{K_t} h dt$. This means that the action integral can be expressed as

$$(11.8) \quad A(u, K) = \int_{t_1}^{t_2} \int_{K_t} H(u, u) h \mu_{K_t} dt = \int_{t_1}^{t_2} \int_{K_t} H(u, k) \mu_{K_t} dt = \int_{t_1}^{t_2} H(K_t, k) dt.$$

In the particular case of the Schwarzschild solution, this leads immediately to

$$(11.9) \quad A(u, K) = \mathcal{M} \Delta t,$$

with the Killing vector normalized so the redshift factor is 1 at infinity.

We can now see that the real problem is the fact that in integrating over a spatial slice, the proper time is elapsing at different rates at different parts of space, so that in general, the quasi-local mass definitions have to contend with this problem whether they like it or not [32], [33]. Thus, in general, if we have no Killing vector field, if t is an arbitrary "time" function on U , and if $K(t_1, t_2)$ is the submanifold of U given by $t_1 \leq t \leq t_2$, then we should simply think of $A(u, K(t_1, t_2)) = \mathcal{M}_{av} \Delta t$, where now \mathcal{M}_{av} is the average quasi-local mass over the given time interval. This naturally leads to taking

$$(11.10) \quad \mathcal{M}(t) = \frac{d}{dt} A(u, K(t_1, t)),$$

as the mass at time t . Thus for the Schwarzschild solution we now find that the mass is \mathcal{M} , the mass parameter, which indicates that the mass parameter is the total mass including that due to gravitational energy.

Another approach to an invariant treatment of mass in general relativity might be based upon the negative of S from (2.10), and the fact that its restriction A_u^{geo} as a linear transformation of $u^\perp \subset T_m M$ has as its eigenvalues the negatives of the principal sectional curvatures which are then the principal tidal accelerations. The eigenvector of the maximum eigenvalue for an observer at $m \in M$ with velocity u then picks out a spatial tangential direction in u^\perp which should be either towards or away from any larger than average matter concentration at locations other than m . For instance in the Schwarzschild solution, it picks out the radial direction, and the maximum time-sectional curvature is $2\mathcal{M}/r^3$. Integrated around a central sphere of radial coordinate r gives therefore $8\pi\mathcal{M}/r$, which is obviously related to the Newtonian potential of the observer located at radial coordinate r in the Schwarzschild gravitational field. This seems to indicate that there might be a way to obtain a generalization of a Newtonian type of potential from the curvature tensor in the form of $-S$.

12. GRAVITATIONAL RADIATION

Frank Tipler has pointed out that due to the definition of the energy momentum stress tensor of the gravitational field in terms of equation (6.1), it follows that the speed of sound in the gravitational field equals the speed of light, for a vacuum electromagnetic field. More specifically, he points out that the gravitational field energy density tensor equals the electromagnetic energy density tensor exactly, according to (6.1) as the contraction of the latter is zero. Thus the speed of sound in the gravitational field due to a vacuum electromagnetic field is equal to the speed of sound in a vacuum electromagnetic field, which is of course the speed of light. This certainly seems reasonable given our way of viewing the gravitational energy density in terms of electromagnetic fields. On the other hand, in the vacuum there is no energy of the gravitational field, and consequently from this point of view, a gravitational wave carries no gravitational field energy through the vacuum.

This point of view has been elaborated previously [7], in what has become known in the literature as the Cooperstock hypothesis, purely on mathematical and somewhat philosophical grounds that the equations for the various pseudo tensors have no content in the vacuum, and as well, on the basis of his detailed computation [7] involving an example of a capacitor in a gravitational wave. On the other hand, the energy

density tensor of the gravitational field is not divergence free which means it can dissipate in one place and appear in another. That is, the time varying matter tensor causes gravitational energy to disappear into the vacuum and then reappear elsewhere where there is matter. This of course is a difficulty for analyzing gravitational radiation, and it means that using a coordinate conserved pseudo-tensor or any other device which is conserved in some useful sense is certainly justified if it aids in calculation.

For instance, Hayward [23], in analyzing gravitational radiation in a quasi-spherical approximation defines an energy density tensor for the gravitational radiation which carries positive energy and in the second approximation reacts on the solution when included in the source of the truncated Einstein equation. This means that one can in special circumstances use special definitions in a way that can be usefully interpreted physically, even if it is technically a fiction. On the other hand, to quote [4], "At the present time there are many solutions of the gravitational wave problem, but none of them are satisfactory...another difficulty: there is no general covariant d'Alembertian, which being in its clear form, could be included into the Einstein equations."

We must keep in mind here, that our view of the energy density of the gravitational field is in complete agreement with the Einstein equation, so it cannot contradict any of its results and likewise, no result of solving the Einstein equation can possibly contradict our view of the energy density of the gravitational field. In particular, both Carl Brans and Frank Tipler (in personal communication) have expressed concerns about how the view expressed here on the gravitational energy momentum stress tensor relates to the analysis of the energy dissipation from binary pulsars, an issue also addressed in [7] in relation to the Cooperstock hypothesis. Particularly relevant here are the calculations in [8] and [9] of the gravitational radiation due to a rotating rod, showing the general relativistic calculation to be consistent with the Cooperstock Hypothesis. The idea that gravitational radiation carries energy away may be a useful idea for keeping track of the various "energies", or conserved quantities, in the system, but the calculations always involve a choice of reference background metric which produces the apparent "energy". Alternately, it seems that there is no mathematical vacuum in realistic models of the universe, because of background radiation and possibly dark energy, so there is background matter to carry the gravitational energy. Since the gravitational energy is really $3p_u/c^2$ in ordinary units, it is so small, that it should be easily carried by the background matter energy in realistic models involving ordinary pressures.

13. BLACK HOLES

Since the vacuum has no energy density, it follows that the assignment of mass to black holes or to cosmological solutions is heavily influenced by boundary conditions assumed for the solution to the Einstein field equations (see e.g. [37]). For instance, in the case of a Schwarzschild black hole, if we try to integrate over a region enclosed by a sphere, we find that it is not the boundary of any compact spacelike slice. The preceding analysis leading to (11.7) would have to be modified to include also an inner boundary as a cutoff so that the region bounded is compact. On the other hand, as it stands, for the Schwarzschild case, the mass is \mathcal{M} no matter what matter resides in the interior as long as the matter is not all inside the Schwarzschild radius, which indicates that it is reasonable to assign the artificial mass \mathcal{M} to the Schwarzschild black hole with mass parameter \mathcal{M} , as a reflection of a boundary condition, the boundary being the black hole horizon. Thus, in the general black hole case, one of the various definitions of quasi-local mass must be adopted. As far as we can see, the actual energy momentum stress tensor of the gravitational field cannot help here.

14. COSMOLOGICAL MODELS

Because the total energy density is simply $(1/4\pi G)Ric$, in any cosmological model where we have a universal time function it is often straight forward to calculate the total energy density which thus includes that of the gravitational field. If the model is specified by a fluid where ρ is the density and p is isotropic pressure (the average of the principal pressures) observed by the universal observer, then $\rho + 3p$ is then the total energy density including that of gravity as seen by the universal observer. Integrating this over the spatial slice at time t , if it is compact, gives the total mass of the model including that due to the gravitational field itself. In fact, it has long been realized that $\rho + 3p$ is the actual source of gravity in

general relativity in many special cases and in particular, in [18], we find $\rho + 3p$ referred to as the "active gravitational mass" of any cosmological fluid model. This seems to have been clear right from nearly the beginning of general relativity [46], so it is rather strange that the energy density tensor of the gravitational field was not realized right from the Einstein equation itself, as soon as the equation was accepted. That is, if gravitational energy has itself effective gravitational mass, then it must be what is accounting for the extra effective gravitational mass over and above the ordinary mass.

15. GENERALIZATIONS

Our treatment of the Einstein equation depends only on assuming that there is a Lorentz manifold of dimension $n + 1$ which describes the spacetime model of the universe and that there is a symmetric tensor field on the spacetime which describes all matter and fields other than gravity together with the EHNIL at each point of the spacetime manifold. However, in our calculations, we assumed in addition that the constant k_n was a universal constant, not depending on the particular event $m \in M$. We can note that this assumption, though natural, is not implied by the principle of relativity, so our arguments without this assumption immediately give the more general equation $Ric - (1/(n - 1))Rg = 2k_n fT$, where f is a smooth function on M . This is because we can still appeal to the principle of relativity to guarantee that all observers at a particular $m \in M$ would see the same gravitation constant, which could then depend on m . The observer principle still applies here, as it applies at each point of M . Using a modification of the Einstein-Hilbert Lagrangian gives the Einstein equation together with an equation for the scalar function f . In spacetime dimension 4, this is usually called Jordan-Brans-Dickie scalar tensor theory [31]. We will keep the constant k_n by giving f the value 1 at our location. We see that this amounts to replacing the source T by a new source fT .

More generally, Moffat [34] has advocated a scalar vector tensor theory (SCVT) in order to solve the dark matter problem. The view in [34] is that SCVT is a new theory of gravity which solves the dark matter problem (he views the dark energy problem as solvable with inhomogeneity [35]). But in fact, a Lagrangian method is used to determine field equations which can be written in the form $Einstein = 8\pi T$, where T is the total energy momentum stress tensor of all ordinary matter and fields as well as that due to the scalar field and that due to the vector field and an additional symmetric tensor. All these extra structures are required to satisfy equations developed by Lagrangian methods, and the free parameters can be chosen to match the dark matter galaxy rotation curves.

However, we can also view these results as being a specific model for the dark matter within Einstein's theory of gravitation. Thus, the equations in [34] for the extra scalar and vector fields of SCVT can be viewed as the beginning of the theory of dark matter instead of a way of doing away with dark matter. On the other hand, Frank Tipler [45] has argued that such extreme measures are not needed and that standard physics may be used to account for the dark matter and energy, as well as several other problems in cosmology. In any case, it is certainly of interest that SCVT explains the galaxy rotation curves, explains the galactic lensing data, explains the bullet cluster data as well as globular clusters within galaxies, explains oscillations in the matter power spectrum, all without any additional dark matter, but fails to explain the pioneer deceleration data as these two spacecraft are reaching the outer parts of our solar system. In addition, in SCVT, apparently black holes and singularities do not exist as there are solutions without event horizons. Here, the big bang is replaced by the universe spontaneously arising from Minkowski space.

In higher dimensions, the usual mathematical method of looking for the form of the equation of gravity fails to give a unique result, but rather introduces many free parameters giving a family of gravitation theories known as Lovelock gravity theories [29], [42]. A reading of [29] shows that the theories were discovered without the aid of Lagrangian methods, but subsequently a Lagrangian was found. This shows that without some physics, neither general mathematics nor Lagrangian methods are capable of arriving at a definitive equation. In addition to these theories of gravity, we now have brane-world theories [43], [20], [21], and $f(R)$ gravity theories [2]. Again, these fall into the general scheme of $Einstein = k_n S$, and can therefore be rewritten in the form $Ric - (1/(n - 1))Rg = 2k_n T$, with T a simple linear combination of S and its contraction (in case of $f(R)$ we can expand f in power series and the linear terms inside the Lagrangian give Einstein's equation when the other resulting terms are moved to the source side of the equation).

Of course, there are additional fields in these theories which are required to satisfy additional equations derived by Lagrangian methods, and which can be thought of as generating new forms of matter. In the case of brane-world models for our universe, the universe we live in is modeled as a Lorentz 4-submanifold of a higher dimensional Lorentz manifold. In the end, usually effective equations for the 4-submanifold are found which means again we can view the result as Einstein's equation with new forms of matter, as well as a higher dimensional Einstein equation for the bulk.

Thus, in view of Theorem 7.1, our equation for gravity can be viewed as being much more general than in the usual view. That is, almost all classical gravity theories are really systems of equations of which Einstein's equation in form is the the most important part of the system. Thus we are inclined toward the view that for any Lorentz manifold model of a universe, (7.5) should be viewed as the energy momentum stress tensor of the gravitational field, $(1/2k_n)Ric$ the total energy momentum stress tensor, and $(1/2k_n)[Ric - (1/(n-1))Rg]$ should be viewed as the geometrically effective total energy momentum stress tensor of matter and fields other than gravity.

We can also note here that the Einstein-Hilbert Lagrangian method gives the Einstein tensor as the geometric side of the equation in any dimension of spacetime, whereas it is only in spacetime dimension 4 that the Einstein tensor coincides with our geometrical tensor $(1/2k_n)[Ric - (1/(n-1))Rg]$. Thus, the Einstein tensor results from the Lagrangian method in all dimensions because the coefficient $1/2$ in the term $(1/2)Rg$ results from the derivative of $\sqrt{-det(g)}$ with respect to $det(g)$ which must be carried out in the variation of the Einstein-Hilbert term of the Lagrangian, no matter the dimension of spacetime. Moreover, the fact that conservation of energy makes scalar curvature constant in higher dimensions indicates that we may be seeing spacetime as 4 dimensional because that is where everything of interest is happening, even in higher dimensional theories. As Norbert Reidel has suggested, maybe the only interesting spacetime dimensions are 4 and infinity.

16. CONCLUSION

The idea that the gravitational field energy can be localized is not in contradiction of Einstein's equation for gravity, but rather in fact is a consequence of it. As soon as we observe the mathematical fact that the Ricci tensor is giving the negative flat Euclidean spatial divergence of each observer's spatial infinitesimal tidal acceleration field, it follows that $(1/4\pi G)Ric$ is the total energy momentum stress tensor of matter fields and gravity, and consequently $T - c(T)g$ must be the energy momentum stress tensor of the gravitational field. That is, the Einstein equation is really an infinitesimal law of gravity governing tidal acceleration which is only a slight correction to the geometric form of Newton's law for tidal acceleration. We say merely a slight correction since the correction is only three times the isotropic pressure, $3p_u$ which in terms of mass density in terrestrial terms is $3p_u/c^2$. But these corrections lead directly and purely mathematically to the Einstein equation for gravity and the vanishing of the covariant divergence of the energy momentum stress tensor of all fields other than gravity.

What this means is that the energy momentum stress tensor of the gravitational field cannot have zero divergence unless spacetime has constant scalar curvature. This also means that the gravitational field gets its energy from ordinary matter and fields other than gravity, confirming the Cooperstock hypothesis [7]. In fact, we are going further than the Cooperstock hypothesis in that we claim the gravitational field energy density would even vanish in pressureless dust, for any observer moving with the dust. This view seems to be dictated by the Einstein equation itself. Moreover, if this view is used on already solved problems and elementary examples and problems, it should lead to new perspectives.

Additionally, viewed in terms of infinitesimal tidal acceleration, it seems that each observer sees the effective gravitational mass density as the inertial mass density plus three times the isotropic pressure, a possibly testable result. In particular, for pure electromagnetic radiation, the effective gravitational mass density is twice the inertial mass density, which could possibly be tested in an inertial confinement fusion laboratory.

Finally, our developments are so general as to be able to include many theories of gravity within the Einstein equation, including higher dimensional theories. This leads naturally to the point of view that these theories are really theories involving new forms of matter within Einstein's theory of gravity. As well,

in case of higher dimensions, it appears the assumption of the vanishing divergence of the energy momentum stress tensor for all non gravitational fields implies that the scalar curvature of spacetime must be constant.

We can view spacetime itself as the gravitational field, so the matter is a disturbance of spacetime causing the gravitational field to have energy momentum stress tensor $T_g = T - c(T)g$. This equation is equivalent to $T = T_g - (1/3)c(T_g)g$. Might not there be a dual concept of matter field such that the failure of spacetime to curve so as to be Ricci flat gives the gravitational energy stress tensor T_g which in turn causes the matter field to have energy momentum stress tensor $T = T_g - (1/3)c(T_g)g$? Thus, above we have attempted to give a physically intuitive way to see how the energy of the gravitational field arises from ordinary energy. Is there, dually, a physically intuitive way to view ordinary energy as arising from gravitational energy?

17. ACKNOWLEDGEMENTS

I am very deeply indebted to Frank Tipler for many extremely useful and helpful conversations and in particular, for making me aware of the historic problems surrounding the energy density of the gravitational field, as well as the recent developments in quasi-local mass. I was unaware of these problems when I developed these arguments for the energy momentum stress tensor of the gravitational field and the resulting derivation of the Einstein equation back in the summer of 2003. I am also indebted to Frank Tipler for encouraging me to write down the details of these arguments. I thank Toni Eastham and Juliette Dupre for helpful conversations and questions. I would also like to thank Norbert Riedel, James Glazebrook, and Emma Previato whose comments and encouragement also lead to improvements in the manuscript. I have benefited from communications with Fred Cooperstock in the final stages of preparation of this manuscript, and thank him for playing Devil's advocate as well as making several very useful comments. Finally, I would also like to sincerely thank an anonymous referee whose ideas have lead to several improvements in the exposition. Of course any errors or controversial views in what remains are entirely due to the author.

18. APPENDIX: SYMMETRIES OF THE RIEMANN CURVATURE TENSOR

We have in (2.6) made the definition

$$\mathcal{K}(v, w)z = \mathcal{R}(z, v)w$$

so as to define the linear transformation $\mathcal{K}(v, w)$ for any tangent vectors $v, w \in T_m M$, and which in particular gives us the geometric infinitesimal tidal acceleration. We claimed above in (2.7) that

$$\mathcal{K}(v, w)^* = \mathcal{K}(w, v).$$

Here m is a fixed event in M . Now the basic symmetries of the curvature tensor [38] give us

$$(18.1) \quad \mathcal{R}(v, w) = -\mathcal{R}(w, v),$$

$$(18.2) \quad g(\mathcal{R}(v, w)x, y) = g(\mathcal{R}(x, y)v, w),$$

and

$$(18.3) \quad g(\mathcal{R}(v, w)x, y) = -g(\mathcal{R}(v, w)y, x).$$

Obviously, we can see that (18.3) is not really fundamental, as it is an immediate consequence of (18.1) and (18.2).

So now, using (18.1), (18.2), and (18.3), we have for any $v, w, x, y \in T_m M$,

$$g(\mathcal{K}(v, w)^*x, y) = g(x, \mathcal{K}(v, w)y) = g(x, \mathcal{R}(y, v)w) =$$

$$g(\mathcal{R}(y, v)w, x) = g(\mathcal{R}(v, y)x, w) = g(\mathcal{R}(x, w)v, y) = g(\mathcal{K}(w, v)x, y).$$

Thus we have shown for any given vectors $v, w \in T_m M$ we have for all vectors $x, y \in T_m M$ that

$$g(\mathcal{K}(v, w)^*x, y) = g(\mathcal{K}(w, v)x, y).$$

Therefore (2.7) holds:

$$\mathcal{K}(v, w)^* = \mathcal{K}(w, v)$$

for all vectors $v, w \in T_m M$.

It is because of (2.7) that we immediately find $\mathcal{K}(u, u)$ is self-adjoint with respect to g , the metric tensor. We thus seem to be able then to easily make the connection between the geometric infinitesimal tidal acceleration at $m \in M$ given through the equation of geodesic deviation and the Newtonian tidal acceleration given by the derivative of the Newtonian gravitational force per unit mass vector field at the point $0 \in T_m M$.

19. APPENDIX: GENERAL PRINCIPLE OF ANALYTIC CONTINUATION

Suppose that E_1, E_2, \dots, E_n , and F are all vector spaces (possibly infinite dimensional). The function or mapping

$$A : E_1 \times E_2 \times \dots \times E_n \longrightarrow F$$

is a multilinear map provided that it is linear in each variable when all others are held fixed. In this case, we say that A is a multilinear map of rank n . A useful notation here is just to use juxtaposition for evaluation of multilinear maps, so we write

$$A(v_1, v_2, \dots, v_n) = Av_1 v_2 \dots v_n$$

whenever $v_k \in E_k$ for $1 \leq k \leq n$. Thus, we simply treat the multilinear map A as a sort of generalized coefficient which allows us to multiply vectors, and the multilinear condition simply becomes the distributive law of multiplication.

In case that $E_k = E$ for all k , there is really a single vector space providing the input vectors, and $A : E^n \longrightarrow F$. We say that A is a multilinear map of rank n on E in this case, even though in reality, the domain of A is the set E^n . Here it is useful to write $v^{(k)}$ for the k -fold juxtaposition of v 's. Thus we have

$$A(v, v, \dots, v) = Av^{(n)}.$$

More generally, then for any positive integer m and vectors $v_1, v_2, \dots, v_m \in E$ and non-negative integers k_1, k_2, \dots, k_m satisfying $k_1 + k_2 + \dots + k_m = n$, we have the equation

$$Av^{(k_1)} v^{(k_2)} \dots v^{(k_m)} = A(v_1, \dots, v_1, v_2, \dots, v_2, \dots, v_m, \dots, v_m)$$

where each vector is repeated the appropriate number of times, v_1 being repeated k_1 times, v_2 repeated k_2 times and so on. Of course, if $k_i = 0$ then that merely means that v_i is actually left out, so $v^{(0)} = 1$ in effect.

We say that $A : E^n \longrightarrow F$ is symmetric if $Av_1 v_2 \dots v_n$ is independent of the ordering of the n input vectors. Thus when dealing with algebraic expressions involving symmetric multilinear maps as coefficients, the commutative law is in effect.

Given any multilinear map A of rank n from E to F we can define a function $f_A : E \longrightarrow F$ by the rule

$$f_A(v) = Av^{(n)}.$$

If we also assume that A is symmetric, then have for any $v_0, v_1, \dots, v_m \in E$,

$$(19.1) \quad f_A(v_0 + v_1 + \dots + v_m) = \sum_{[k_0+k_1+\dots+k_m=n]} C(n; k_0, k_1, \dots, k_m) Av_0^{(k_0)} v_1^{(k_1)} \dots v_m^{(k_m)}.$$

Here $C(n; k_0, k_1, \dots, k_m)$ is the multinomial coefficient:

$$(19.2) \quad C(n; k_0, k_1, \dots, k_m) = \frac{n!}{k_0! k_1! \dots k_m!}.$$

First suppose that the rank n symmetric multilinear map A has the property that $f_A : E \longrightarrow F$ is constant as a function on E . Let $v_0, v_1, \dots, v_n \in E$. Notice we are here dealing with the case that $m = n$. There are possibly $n + 1$ different vectors here. Notice that the term on the right-hand side of (19.1) with $k_0 = n$ is simply $Av_0^{(n)} = f_A(v_0)$. Also notice that the term $k_0 = 0$ must have $k_1 = k_2 = \dots = k_n = 1$, which

gives $n!Av_1\dots v_n$. Let $w = v_0 + v_1 + \dots + v_n$. Since we assume that f_A is constant on all of E , it follows that $f_A(w) = f_A(v_0)$ and therefore by (19.1), for all vectors $v_0, v_1, \dots, v_n \in E$, we have

$$(19.3) \quad n!Av_1v_2\dots v_n + \sum_{[k_0+k_1+\dots+k_m=n, 0 < k_0 < n]} C(n; k_0, k_1, \dots, k_m) Av_0^{(k_0)} v_1^{(k_1)} \dots v_m^{(k_m)} = 0.$$

Notice that v_0 can now be taken equal to 0 in equation (19.3) and the result is $n!Av_1v_2\dots v_n = 0$, for any vectors $v_1, v_2, \dots, v_n \in E$. This means that $A = 0$. We have therefore proven

Proposition 19.1. *If A is a symmetric multilinear map on a vector space E with values in the vector space F and if the function $f_A : E \rightarrow F$ is constant on E , then $A = 0$.*

Suppose now that E is a topological vector space and that U is a non-empty open subset of E on which f_A is constant. Let $v_0 \in U$ and let $w_1, w_2, \dots, w_n \in E$ be arbitrary. Because U is open in E and the operations of vector addition and scalar multiplication are continuous mappings, it follows that there is a positive number ϵ so that if t_1, t_2, \dots, t_n are any numbers with $|t_i| \leq \epsilon$, $1 \leq i \leq n$, then

$$v_0 + t_1w_1 + t_2w_2 + \dots + t_nw_n \in U.$$

If we now take $v_i = t_iw_i$ for $1 \leq i \leq n$ in our previous calculation (19.3), we find that if $|t_i| \leq \epsilon$, $1 \leq i \leq n$, then

$$(19.4) \quad n!(t_1t_2\dots t_n)Av_1v_2\dots v_n + \sum_{[k_0+k_1+\dots+k_m=n, 0 < k_0 < n]} (t_1^{(k_1)} \dots t_n^{(k_n)}) C(n; k_0, k_1, \dots, k_m) Av_0^{(k_0)} w_1^{(k_1)} \dots w_m^{(k_m)} = 0.$$

In particular, this means that

$$(19.5) \quad n!\epsilon^n Av_1v_2\dots v_n + \sum_{[k_0+k_1+\dots+k_m=n, 0 < k_0 < n]} \epsilon^n C(n; k_0, k_1, \dots, k_m) Av_0^{(k_0)} w_1^{(k_1)} \dots w_m^{(k_m)} = 0.$$

and therefore

$$(19.6) \quad n!Av_1v_2\dots v_n + \sum_{[k_0+k_1+\dots+k_m=n, 0 < k_0 < n]} C(n; k_0, k_1, \dots, k_m) Av_0^{(k_0)} w_1^{(k_1)} \dots w_m^{(k_m)} = 0.$$

But then, by (19.1) and (19.6), we have $f_A(v_0) = f_A(v_0 + w_1 + w_2 + \dots + w_n)$ no matter the choice of vectors $w_1, \dots, w_n \in E$. This means that f_A is constant on E and by Proposition 19.1 we now conclude that $A = 0$. We have now proven

Proposition 19.2. *suppose E is any topological vector space and F is any vector space. Suppose that $A : E^n \rightarrow F$ is any symmetric multilinear map of rank n . If there is a non-empty open subset of E on which $f_A : E \rightarrow F$ is constant, then $A = 0$.*

By convention, a multilinear map from E to F of rank zero is just a vector in F . If A_k is a symmetric multilinear map of E to F of rank k , for $0 \leq k \leq n$, then the function

$$f = \sum_{k=0}^n f_{A_k}$$

is a polynomial function of degree n . If F is also a topological vector space, then we can take limits in the sum and consider power series. The general principle of analytic continuation relies on the uniqueness of power series expressions. In general, for Banach spaces, if two power series agree locally as functions, then all their coefficients are the same—that is, they are the same power series. The proof is easy using differentiation, just use the same method used in freshman calculus, but for Banach space valued functions. We have basically proven this fact in case there is only one term in the power series, but without using differentiation and without even having topology on the range vector space.

The Observer Principle as we have formulated it here is just this special case of the principle of analytic continuation given in Proposition 19.2. In a sense, it is the essence of the Principle of Relativity. Because it says that in order for two rank r symmetric tensors A and B on T_mM to agree, $A = B$, it merely suffices that $Au^{(r)} = Bu^{(r)}$ for every observer $u \in T_mM$. A law in general relativity at event $m \in M$ expressed as

an equation of symmetric tensors is true if and only if each observer sees the two tensors as equal. We can think of observer $u \in T_m M$ observing the rank r tensor A on $T_m M$ by actually finding the value of $Au^{(r)}$. For further discussion of this topic, see the Appendix of [12].

20. APPENDIX: COMMENTS OF ENERGY DENSITY OF GRAVITY

We must keep in mind here that our "scooping out" in either the positive pressure case or the negative pressure case is really just a thought experiment in the sense of one of Einstein's "gedanken" experiments. We are just imagining that for an infinitesimal amount of time δt as seen by observer u , that as if by magic, a small part of physical reality were replaced by either laser beams or capacitor as the case may be. Thus, in the negative energy case, we are not asking the observer to sort the charges along the cut line, we are merely asking the reader to imagine what the observer would see "energy-density-wise", if for a certain infinitesimal duration δt his physical reality was magically modified with electromagnetic fields so as to balance the pressures. Notice, that for the laser beam argument, we must assume that the duration is large enough that statistically many photons strike the opposite faces for the creation of photon pressure. This means that $c\delta t$ should be large in comparison to $\delta x, \delta y, \delta z$, but as we are dealing with infinitesimals in the macroscopic sense of physics, this is not a problem.

For more detail in the "pillbox" argument, we must assume that the charge surfaces behave like conductors so that there is no field outside the scooped out region nor parallel to the scooped regions opposite pair of faces which we are considering. This, of course, is somewhat of a stretch. We must assume that the capacitor's electric field makes no change to the region outside. Suppose \mathbf{E} is the electric field vector inside the scooped spatial region and that σ is the surface charge density. By Gauss' Law for electric flux, we then have the electric field flux $|\mathbf{E}|A = A\sigma$, where A is the surface area of one of the faces of the scooped out region across which we have the negative pressure, say $A = \delta y \delta z$, and the separation of two charged surfaces is δx . Thus,

$$|\mathbf{E}| = \sigma.$$

The energy of two such separated charged surfaces is therefore $|\mathbf{F}|\delta x$ where \mathbf{F} is the force exerted by one face on the other. But then $\mathbf{F} = A\sigma\mathbf{E}$ so the energy is

$$|\mathbf{F}|\delta x = \sigma|\mathbf{E}|A\delta x = A\sigma^2\delta x = \sigma^2V,$$

where $V = A\delta x$ is the volume. Thus the energy density equals the square of the surface charge density. On the other hand, the pressure in the x -direction here in absolute value satisfies

$$A|p_x| = |\mathbf{F}| = A\sigma|\mathbf{E}| = A\sigma^2.$$

Therefore, the absolute value of the pressure also equals the square of the surface charge density. Thus, we conclude that the energy density of an electric field inside the scooped region caused by charges on opposite faces attracting each other is exactly the pressure in absolute value.

There are clearly problems with the capacitor argument as far as actually physically putting it into effect is concerned. For instance, the edge effect of a finite parallel plate capacitor which causes electric field lines to "bulge out" is certainly undesirable, and the argument would be ruined by the electric fields created outside the scooped out region. Somehow, the fact that the energy density equals the pressure in both the positive and negative pressure cases here, however, seems to be too much of a coincidence to ignore. This seems to indicate that there must be some very strong connection between gravity and electromagnetism. For instance, one might imagine that a whole space-like slice of spacetime is chopped up into such tiny infinitesimal bits and everything replaced with such infinitesimal electromagnetic field systems. How would the observers know? All the pressures they feel would be the same everywhere. I would love to hear comments from physicists on this.

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