

The Einstein equation and the energy density of the gravitational field

Maurice J. Dupré
Department of Mathematics, Tulane University
New Orleans, LA 70118 USA
mdupre@tulane.edu

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Abstract

We give a derivation of the Einstein equation for gravity which employs a definition of the local energy density of the gravitational field as a symmetric second rank tensor whose value for each observer gives the trace of the spatial part of the energy-stress tensor as seen by that observer. We give a physical motivation for this choice using light pressure.

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1 INTRODUCTION

Since Einstein's and Hilbert's original "derivations" of the Einstein equation for gravity in classical general relativity (CGR), there have appeared too many to list. The many different types of derivations are summarized in [15]. In fact, all these subsequent derivations as well as Hilbert's original derivation contrast markedly from Einstein's original derivation in that they appeal to some abstract mathematical principle which though desirable, is usually not justifiable beyond mere desire. For instance, one of the most popular derivations simply modifies one side of the equation to make it have zero divergence on grounds that physical considerations make the other side, the energy-stress tensor, have zero divergence. On the other hand, in Einstein's original derivation, [4], we see the realization that mathematically the Ricci tensor should be proportional to the source which should be the total energy density due to both the energy-stress tensor as well as the gravitational field itself. However, in [4], Einstein was not able to arrive at a tensor expression for the energy density of the gravitational field. Instead, he arrived at a doubly indexed quantity which he admitted was not a tensor, but rather a pseudo-tensor defined in terms of the connection coefficients and which served to give the energy of the gravitational field for purposes of deriving the equation. As the arguments in [3] leading up to the development of CGR show, Einstein was clearly thinking of the energy of the gravitational field in a Newtonian way, since in particular, the connection coefficients are the generalized gravitational forces from the Newtonian viewpoint. Moreover, in [4], Einstein was very clear that his equation was using the energy density

of the gravitational field in addition to the energy-stress tensor as the total source of gravity. In fact, subsequent attempts to mathematically characterize the energy of the gravitational field have all basically clung to the Newtonian framework which makes the energy of the gravitational field a function of a non-local arrangement of masses and energies. So much so, that these views are now taken for granted to the point that in [9] we have the claim of the impossibility of existence of a local energy density tensor for the gravitational field (see also [17],[6],[5],[2],[16]). This attitude clearly persists to the present as expressed, for instance, in chapter 3 of [16]. The result has been a profusion of mathematically inspired notions of quasi-local mass, which all have their advantages and drawbacks as discussed in [16] and [8], along with extremely involved analysis required to arrive at their basic properties. Fortunately, I was unaware of these problems when I set out to find the energy density of the gravitational field in order to derive Einstein's equation. By contrast, we can realize the physical representation of the local energy density of the gravitational field by thinking of the physics of the gravitational field in relativistic terms. When we do, we see that the resulting second rank tensor added to the energy-stress tensor can serve as the total energy density source for the gravitational field which then must be proportional to the Ricci tensor. The result in particular immediately gives the Einstein equation. The trick is to adopt a truly relativistic attitude towards what disturbance of the gravitational field entails, and using laser light pressure as a standard, to relate the disturbance back to the energy-stress tensor itself. In particular, we find that the divergence of the energy-momentum-stress tensor due to matter and fields other than gravity must be zero as a consequence of our derivation.

2 THE RICCI TENSOR AND DIVERGENCE

We assume that our spacetime is a 4-manifold M equipped with a Lorentz metric tensor g , with signature $(-, +, +, +)$, and we denote by ∇ the resulting Kozul connection or covariant differentiation operator on M . We use $T_m M$ to denote the tangent space to M at $m \in M$. It is convenient in this setting to refer to $u \in T_m M$ as a unit vector to mean merely $|g(u, u)| = 1$. We have then the Riemann curvature operator, \mathcal{R} , given by

$$\mathcal{R}(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u, v]}, \quad (2.1)$$

where u and v are any tangent vector fields on M . We note that $\mathcal{R}(u, v)$ actually defines a vector bundle map of the tangent bundle TM to itself covering the identity map of M , and it as well then determines the Riemann curvature tensor, R , of fourth rank, which means that \mathcal{R} is itself a linear transformation valued tensor field on M . One of our main concerns is the certain contraction of R known as the Ricci tensor, Ric . In fact in any frame at $m \in M$ with basis e_α for $T_m M$ and dual basis ω^α , We have, using the summation convention,

$$Ric(u, v) = \omega^\alpha(\mathcal{R}(e_\alpha, u)v), \quad u, v \in T_m M. \quad (2.2)$$

Among the many symmetries of the Riemann curvature tensor is the fact that Ric is a symmetric tensor.

In order to see how the Ricci tensor enters into the theory of gravity, we should recall the equation of geodesic deviation. If $[-a, a]$ and $[-b, b]$ is a pair of intervals in \mathbb{R} , then a Jacobi field is a smooth map $J : [-a, a] \times [-b, b] \longrightarrow M$ such that for each fixed $s \in [-a, a]$ the map

$J_s; [-b, b] \rightarrow M$, given by $J_s(t) = J(s, t)$, is a unit speed geodesic in M . We can then form local vector fields e, u on an open neighborhood of the image of J in M , denoted $Im J$, so that

$$e(J(s, t)) = \partial_s J(s, t), \quad u(J(s, t)) = \partial_t J(s, t). \quad (2.3)$$

Thus we must have $[e, u] = 0$ and $\nabla_u u = 0$, on $Im J$, so we find

$$\mathcal{R}(e, u)u = -\nabla_u \nabla_e u. \quad (2.4)$$

We will call e in this situation a tangent Jacobi field along J_0 . In fact, given m a point on J_0 and any unit vector $e_m \in T_m$ which is orthogonal to $u(m)$, we can arrange that $e(m) = e_m$. Since our connection is assumed to be the unique torsion free metric connection, we have $[e, u] = \nabla_e u - \nabla_u e$, so the condition that $[e, u] = 0$ gives $\nabla_e u = \nabla_u e$ in our present case. In view of (2.4), we then find the equation of geodesic deviation on $Im J$.

$$\mathcal{R}(e, u)u + \nabla_u \nabla_e u = 0. \quad (2.5)$$

Now the term $\nabla_u \nabla_e u$ should be interpreted as the rate of change of separation acceleration in direction e of infinitesimally separated geodesics. If δs is a small change in the parameter s , then we can think of $(\delta s)e$ as representing the separation between the geodesic J_0 and the geodesic $J_{\delta s}$. Thus $\nabla_u [(\delta s)e]$ represents the rate of change of separation from $J_{\delta s}$ as seen by an observer moving along J_0 . Then $\nabla_u \nabla_u [(\delta s)e]$ represents acceleration in separation between the observer following J_0 and the geodesic $J_{\delta s}$. Dividing by δs then gives the rate of change in the direction e of the separation acceleration. This means that $g(e, \nabla_u \nabla_e u) = g(e, \mathcal{R}(e, u)u)$ is a "term" in a typical spatial divergence calculation, in this case of the acceleration field. That is, it is the e -component of the rate of change of separation acceleration in direction e . If u is assumed time-like, it follows from (2.2) that $Ric(u, u)$ is in fact giving an invariant form of a spatial divergence of the separation acceleration field for infinitesimally close geodesics to J_0 , as would be seen by an observer following along J_0 . To be clearer about this, an observer following J_0 could view the separation of nearby geodesics as a position vector field on u^\perp , the orthogonal complement of his velocity vector, so the change in separation would then be viewed as the spatial velocity of nearby points (test particles suspended) in his space, so its rate of change is the acceleration field \mathbf{a}_u as seen by the observer. We then have for $e \in T_m M$ that

$$\mathbf{a}_u(e) = \nabla_u \nabla_e u, \quad e \in u^\perp \subset TM. \quad (2.6)$$

If (e_1, e_2, e_3) is a smooth frame field along J_0 for the orthogonal complement of the time-like unit vector field u along J_0 , which is orthonormal at m on J_0 , then from (2.2) and (2.6), we have at m ,

$$Ric(u, u) = -[g(e_1, \nabla_u \nabla_e u) + g(e_2, \nabla_u \nabla_e u) + g(e_3, \nabla_u \nabla_e u)] = -div_u \mathbf{a}_u, \quad (2.7)$$

where \mathbf{a} denotes the spatial separation (compare [10], 8.9, page 219 and [15], 4.2.2, page 114) acceleration field around J_0 , and $div_u \mathbf{a}$ denotes the spatial divergence of this spatial vector field around J_0 . At this point, one might object that the observer could be rotating which would introduce fictional acceleration into \mathbf{a}_u , and that is correct. A more sophisticated analysis here could deal with this purely mathematically (see for instance [5], [9], or [15]), but let us allow that the observer can feel if he is rotating and just say he restricts to cases where he is not rotating in order to carry out his measurements. Continuing then, for a non-rotating observer, the separation acceleration

field in a geometric theory of gravity is the essence of the gravitational field. That is, if an observer at event $m \in M$ has velocity u , with $g(u, u) = -1$, then according to (2.5), (2.6), and (2.7) we should interpret $R(u, u)$ as the negative divergence of the gravitational "force per unit mass" field as seen by that observer at $m \in M$. Now, in ordinary vector analysis, the divergence of an irrotational vector field, divided by 4π , is its source density, and we know that the source density for gravity is energy density. Therefore there should be a universal positive constant G so that $(1/4\pi)Ric(u, u)$ is equal to the energy density observed by this observer multiplied by G , as gravity always acts as an attractive influence between objects. But, thinking relativistically, if the gravitational field itself has energy, that energy density must be included in the source total energy density. Thus, the question now arises as to the energy density of the gravitational field itself.

3 THE ENERGY DENSITY OF THE GRAVITATIONAL FIELD

In order to deal with the energy density of the gravitational field, we must first think in terms of the basic assumption of the geometric notion of gravity which is that "free test" particles must follow geodesics. If this is the case, then from the point of view of the gravitational field itself, it is happiest when all particles are following geodesics. In fact, we can imagine that in a limiting sense, if "all particles" follow geodesics, then the gravitational field is completely relaxed and contains no energy. It is only when we try to push a particle off of its geodesic that we feel the reaction of the gravitational field, and notice we feel it right at the location of the event of trying to push the particle off of its geodesic. Thus, relativistically, we should think of the manifestation of tension in the gravitational field is particles not following geodesics. Now, if a particle is not following a geodesic, then it is because it is being acted on by a force which is not part of the gravitational field itself. Because by definition, gravity acts only through causing particles to follow geodesics, in the absence of "outside" forces. When a force acts to move a particle a certain amount off of its geodesic path, the force required to do so is proportional to the particles inertial mass, by definition, but in essence, this says the gripping energy of the gravitational field at the point where the particle is located is somehow related to the mass of the particle. Accepting this, the force density should be manifested in pressures in various directions. That is, the energy-stress tensor tells us the pressures as seen by any observer in various directions, so from the energy-stress tensor itself, we should be able to find the energy density of the gravitational field. For instance, the pressure you feel on your bottom when sitting in a chair is a manifestation of the energy density of the gravitational field at those points on your chair. In a sense then, we could say that if the surface of your chair were replaced by an infinitesimally thin slab sitting on top of an infinitesimally lower chair, then the mass energy of the slab required to hold you in place divided by the volume of the slab is a reflection of the energy density of the gravitational field. What is the minimum mass which can take care of this job? In fact, the material the chair is made of in some sense is a reflection of the energy density of the gravitational field right where your chair is located. Even primitive people have an intuitive idea of the strength of material needed to make a chair, and thus have a working idea of the energy density of the gravitational field. We should therefore think of the least mass energy of material required to make a chair as a rough measure of the energy density of the gravitational field where the chair is to be used. More generally, imagine an observer located at $m \in M$, ghost-like inside a medium with energy-stress tensor T and suppose that his velocity at m is u . Then exponentiating $u^\perp \subset T_m M$, the orthogonal complement of u in $T_m M$, the pressures are given by the restriction of T_m to $u^\perp \times u^\perp$. This is a symmetric tensor on a Euclidian space so can be diagonalized. Notice

that this does not mean T itself is diagonalizable. Thus, there is an orthonormal frame (e_x, e_y, e_z) for u^\perp with the property that $T(e_a, e_b) = p_a \delta_b^a$, for $a, b \in \{x, y, z\}$. Imagine scooping out a tiny box in M at m whose edges are parallel to these principal axes of this spatial part of the energy-stress tensor. We can imagine putting infinitesimally thin reflecting mirrors for walls of the box and filling the box with laser beams reflecting back and forth in directions parallel to the edges of the box with enough light pressure in each direction to balance the force from outside on these reflecting walls. In a sense, we have standardized a system to balance the pressures acting to disturb the gravitational field, so we define the energy density of the gravitational field as seen by our observer to be the energy density of the light in this little box. The fact that a photon has zero rest mass should mean that the light energy constitutes a minimum amount of energy to accomplish this task of balancing the gravitational energy. However, it is an elementary problem in physics to see that the energy density of the light along a given axis is exactly the pressure in that principal direction. Let us review this simple argument. Assume the coordinates are (t, x, y, z) for simplicity and the box edges are parallel to these axes with lengths $\delta x, \delta y, \delta z$, respectively. Assume that the laser beams parallel to say the x -axis contain N_x photons, each having spatial momentum P_x . In time δt , the photons travel a distance of $c\delta t$ and hence each such photon makes $(c\delta t)/(\delta x)$ reflections for a change in momentum of $2P_x$ for each reflection. Thus the total momentum transfer to the two end walls perpendicular to the x -axis for the laser beams paralleling the x -axis is

$$\frac{2P_x N_x c \delta t}{\delta x}. \quad (3.1)$$

This means that the force exerted on the two end walls is $(2P_x N_x c)/(\delta x)$. But the total area of the two end walls is $2\delta y \delta z$, so the pressure on the end walls is

$$p_x = \frac{N_x P_x c}{V}, \quad (3.2)$$

where $V = \delta x \delta y \delta z$ is the volume of the box. But the relativistic energy of a photon with momentum P_x is $P_x c$. Therefore, the total energy density due to the x -axis beams is exactly the pressure in the x -direction on the walls perpendicular to the x -axis. Likewise for the other two axes, consequently we see that the energy density in the box is the sum of the pressures that the beams are balancing, that is the trace of the observer's spatial part of the energy-stress tensor, $p_x + p_y + p_z$.

4 THE ENERGY DENSITY TENSOR OF THE GRAVITATIONAL FIELD

In this section, and the next, we will be applying what we will call the *symmetric tensor observer principle* (see Appendix on Analytic Continuation). In our situation, as applied to second rank tensors, it means that if A and B are both symmetric second rank tensors at $m \in M$, and if $A(u, u) = B(u, u)$ for all unit time-like vectors, it follows purely mathematically, that $A = B$. This is the essence of the **Principle of Relativity** as applied to second rank symmetric tensors—a law (at m), say $A = B$, should be true for all observers (at m) and conversely, if true for all observers (at m), that is if $A(u, u) = B(u, u)$ for all time-like unit vectors $u \in T_m M$, then it should be a law (at m) that $A = B$. We are going to obtain the Einstein equation by simply observing that if H is the symmetric tensor giving the total energy density at $m \in M$, then $(1/4\pi G)Ric(u, u) = GH(u, u)$

for each vector $u \in T_m M$ such that $g(u, u) = -1$. First, let $c(A)$ denote the contraction of A for any second rank covariant tensor A . Thus using the summation convention, we have

$$c(A) = g^{\alpha\beta} A_{\alpha\beta}. \quad (4.1)$$

Now suppose that $u \in T_m M$ is the velocity of an observer at $m \in M$. Suppose that T is the second rank covariant energy-stress tensor. We can define the projection tensor $P_u : T_m M \rightarrow T_m M$ by $P_u(w) = w + g(w, u)u$, for any $w \in T_m M$. Note here that we are not assuming w is time-like or space-like, it is just any vector in $T_m M$. We easily check that P_u is linear, that $P_u \circ P_u = P_u$, and that $P_u(T_m M) = u^\perp \subset T_m M$. For instance, if $w \in T_m M$, then $g(u, P_u(w)) = g(u, w) + g(w, u)(-1) = 0$, so $P_u(T_m M) \subset u^\perp$. On the other hand, if $w \in u^\perp$, then obviously $P_u(w) = w$, and therefore, $P_u(T_m M) = u^\perp$. Thus, $A(v, w) = T(P_u(v), P_u(w))$ is a symmetric tensor at m and restricted to u^\perp , this tensor can be diagonalized as g on u^\perp is positive definite. That is to say, there is an orthonormal frame (e_1, e_2, e_3) for u^\perp so that $A(e_i, e_j) = \lambda_i \delta_j^i$. Thus, we see that the box argument of the preceding section shows that the sum of these diagonal values of A must represent the energy density of the gravitational field, namely, $a = A(e_1, e_1) + A(e_2, e_2) + A(e_3, e_3)$. Now, if we compute $c(T)$ in the frame (u, e_1, e_2, e_3) , we find that $c(T) = -T(u, u) + a$, and therefore, $a = T(u, u) + c(T) = T(u, u) - c(T)g(u, u)$, which is to say finally that the second rank symmetric covariant tensor T_g , where

$$T_g = T - c(T)g \quad (4.2)$$

represents the energy density of the gravitational field at $m \in M$, because by the symmetric tensor observer principle, T_g at $m \in M$ is completely determined by specifying $T_g(u, u)$ for each time-like unit vector $u \in T_m M$.

5 THE DERIVATION OF THE EINSTEIN EQUATION

Now, finally, from the fact that $(1/4\pi)Ric(u, u)$ is the divergence of the spatial gravitational acceleration per unit mass, using (4.2) our gravitation equation should be that for every timelike unit vector (that is every observer) at $m \in M$, we have

$$\frac{Ric(u, u)}{4\pi} = G[T(u, u) + T_g(u, u)] = G[T(u, u) + T(u, u) - c(T)g(u, u)]. \quad (5.1)$$

Since (5.1) is homogeneous of order 2, it follows that the equation is true for u being any timelike vector, and hence by the symmetric tensor observer principle, we must have

$$\frac{Ric}{4\pi} = G[2T - c(T)g] = 2G[T - (1/2)c(T)g]. \quad (5.2)$$

Let us denote the total energy-stress tensor by $H = T + T_g = 2T - c(T)g$. In these terms, we have simply

$$Ric = 4\pi G H. \quad (5.3)$$

Equivalently, we have

$$Ric = 8\pi G[T - (1/2)c(T)g] \quad (5.4)$$

which is a well-known form of Einstein's equation. As $c(g) = 4$ and $c(Ric) = R$, where as usual, R is the scalar curvature, we find that $R = (8\pi G)[-c(T)]$, so the equation can be also written as $Ric = 8\pi GT + (1/2)Rg$, and this results immediately in the most familiar form of the Einstein equation

$$E = Ric - (1/2)Rg = 8\pi G T, \quad (5.5)$$

where $E = Ric - (1/2)Rg$ is the Einstein tensor. Notice that we have not used local conservation of energy, $divT = 0$. Since the left side of (5.5), the Einstein tensor, E , is divergence free, we find $divT = 0$ as a consequence of our derivation.

We can observe that our derivation required the assumption that the pressures are positive in order for the laser light box argument to justify the expression on the right side of (5.2) as the total energy density tensor of the matter fields and gravitational fields, but the development is so general at this point, that it seems reasonable that the expression (4.2) should be regarded as the energy density of the gravitational field in all cases. This means that the total energy density is the expression H on the right hand side of (5.3),

$$H = T + T_g = 2T - c(T)g = 2[T - (1/2)c(T)g]. \quad (5.6)$$

Reconsidering $T_g = T - c(T)g$, it is probably more natural to think of the gravitational field's energy-stress tensor, T_g as a function of the metric tensor in some way. For this we just use the Einstein equation itself. Since $c(T) = (-1/8\pi G)R$, from (5.5) we immediately conclude that

$$T_g = (1/8\pi G)[Ric + (1/2)Rg] = (1/8\pi G)(E + Rg). \quad (5.7)$$

From the last expression on the right, we see, as T and the Einstein tensor, $E = Ric - (1/2)Rg$ both have zero divergence, that

$$divH = divT_g = (1/8\pi G)div(Rg) = (1/8\pi G)dR = -d[c(T)]. \quad (5.8)$$

So even though the total energy and gravitational energy are not infinitesimally conserved, the divergence is simply proportional to the exterior derivative the scalar curvature. Of course, $divT_g = -d[c(T)]$ is obvious from the definition, (4.2), once we accept $divT = 0$. In particular, as $d^2 = 0$, this means that

$$d[divH] = d[divT_g] = 0, \quad (5.9)$$

but (5.8) is even better as it shows $divT_g$ is an exact 1-form on M . On the other hand, the equation (5.8), when written

$$divT_g + d[c(T)] = 0 \quad (5.10)$$

has another interpretation. In classical continuum mechanics written in four dimensional form of space plus time, the divergence of the energy stress tensor equals the density of external forces. Of course in relativity, the energy stress tensor T contains everything and there are no external forces, as gravity is not a force. But, we can view (5.10) as saying that from the point of view of the gravitational field, the matter and fields represented by T are acting on the gravitational field as an external force density of $-d[c(T)]$. In classical continuum mechanics, the external force density

has zero time component, but relativistically such is not the case, the force only has zero time component in the instantaneous rest frame of the object acted on. We can therefore view (5.10) as saying that the divergence of the the gravitational field's energy stress tensor is being balanced by the rate of increase of $-c(T)$. If p_x, p_y, p_z are the principal pressures in the frame of an observer with velocity u , where $g(u, u) = -1$, then $\rho = T(u, u)$ is the energy density observed, and $divT_g(u)$ is then the power loss density of the gravitational field. Now $c(T) = -\rho + p_x + p_y + p_z$, so (5.10) becomes

$$divT_g(u) = D_u\rho - D_u[p_x + p_y + p_z]. \quad (5.11)$$

Thus, the observer sees the divergence of energy of the gravitational field is exactly the rate of increase of energy density of the matter and fields less the rate of increase of principal pressures. In particular, in any dust model of the universe (pressure zero), the gravitational energy dissipation is exactly balanced by the rate of increase of energy density of the matter and fields. If T is purely the electromagnetic stress tensor in a region where there are only electromagnetic fields, then $c(T) = 0$, and the gravitational energy-stress tensor has zero divergence, so is then infinitesimally conserved.

6 THE GRAVITATION CONSTANT G

So far, we have not said anything about the determination of the gravitation constant G . To evaluate this, we merely need to check the results of experiments with attractive "forces" between masses. But it is much simpler to just use Newtonian gravity in an easy example where the results should be obviously approximately the same. Consider an observer situated at the center of a spherical dust cloud of uniform density ρ , and calculate the separation acceleration field using Newton's law of gravitation. At distance r from the center, but inside the cloud, the mass acting on test particles at radial distance r is simply the mass inside that radius, $M(r)$, by spherical symmetry, as is well-known in Newtonian gravitation. Here, we have $M(r) = (4/3)\pi r^3\rho$. But Newton's Law says the acceleration of a test mass near the center of the dust cloud is radially inward, and if r is the distance from the center, then the radial component of acceleration is given by

$$a_r(r) = -G_N \frac{M(r)}{r^2} = -G_N \frac{4\pi\rho r}{3}. \quad (6.1)$$

Here, G_N is the Newtonian gravitation constant. On the other hand, considering an angular separation of θ , the spatial separation is $s = r\theta$, so the relative acceleration of nearby test particles in the s - direction perpendicular to the radial direction is therefore

$$a_s(r) = \theta a_r(r) = -G_N \frac{4\pi\rho r\theta}{3} = -G_N \frac{4\pi\rho s}{3}. \quad (6.2)$$

Thus the rate of change of separation acceleration of nearby radially separated test particles in the radial direction at given r is by (6.2),

$$\frac{da_r}{dr} = -G_N \frac{4\pi\rho}{3}, \quad (6.3)$$

whereas in the s direction we have the rate of change of separation acceleration is

$$\frac{da_s}{ds} = -G_N \frac{4\pi\rho}{3}, \quad (6.4)$$

the same result again. But there are two orthogonal directions perpendicular to the radial, so now we see that if \mathbf{a}_u denotes the spatial acceleration field around our observer at the center of the dust cloud, then

$$\operatorname{div}_u(\mathbf{a}_u) = -G_N 4\pi\rho. \quad (6.5)$$

As we are dealing with dust, the pressures are zero, so there is no gravitational energy density, and thus ρ is now the total energy density seen by our observer. Thus, we have by (2.7), that $R(u, u) = G_N 4\pi\rho = 4\pi G_N H(u, u)$. But now comparing this result with (5.3), we see that we must have $G = G_N$. Notice that in our development, we have used the symmetric tensor observer principle as a form of the principle of general relativity to reduce everything to working with the time component in an arbitrary frame for the tangent space. The trick is to be able to work completely generally so that conclusions apply to T_{00} and Ric_{00} no matter the coordinate frame, which seems best expressed by using $T(u, u)$ and $R(u, u)$, to remind us that we are dealing with an arbitrary time-like unit vector. It is only now at the end once we have Einstein's equation that we allow a calculation in a special frame in order to evaluate the gravitation constant.

At this point let us discuss for a moment the derivation of Einstein's equation given in [5]. In effect, the derivation of the Einstein equation given in [5] uses the analysis of the special case of a static arrangement of mass for a gravitating fluid drop and adds the Newtonian energy density of the fluid drop as expressed in terms of pressure through the requirement that its surface pressure be zero to get the time component of the Einstein equation. Since the setup is a special arrangement of mass, one cannot assert the symmetric tensor observer principle, because the only observer for which the equation works is the special observer moving with the drop. However, one can appeal to the general covariance desire of relativity that equations should be tensor equations valid in all frames, from which one surmises that if you have found an equation relating the time components in a special frame, then the other components in that special frame should also be equal. Once you accept the full tensor equation in any frame, then it is valid in all frames and you next surmise that if it works for the liquid drop, then it must work in general. But, in our present situation, we have the full equation, and can simply go backwards through the development in [5] to see that the time component of the equation in the liquid drop case is Newton's law, and therefore again conclude that our G in (5.5) is identical to the Newtonian gravitational constant. For a treatment of linearized Einstein gravity and its Newtonian approximation in general, one can consult [9] or [17].

At this point, we can simply choose units such that $G = 1$ and we henceforth drop this factor from the equation for simplicity.

7 ENERGY CONDITIONS

Since we have an expression for the total energy density $H = T + T_g$, we could surmise that in general it would be reasonable to have $H(u, u) \geq 0$, for every time-like tangent vector $u \in TM$. But, this is the same as requiring that $2T(u, u) \geq -c(T)$, a condition known as the *strong energy condition*. In formulating the various energy conditions, it will be useful to denote $\hat{A} = A - (1/2)c(A)g$, when A is any second rank covariant tensor. For instance, we observe easily that $\hat{\hat{A}} = A$, and $\hat{Ric} = E$. We say A is observer non-negative definite if $A(u, u) \geq 0$, for every time-like vector u , whereas we say that A is dominantly non-negative if $A(u, v) \geq 0$ whenever u and v are vectors with $g(u, v) < 0$.

Picking a future half of the light cone arbitrarily at $m \in M$, we note that the future light cone is an open subset of the tangent space $T_m M$, and therefore there is a basis for the tangent space consisting of future time-like unit vectors, say u_1, u_2, u_3, u_4 . In this frame we have $A_{\alpha\beta} \geq 0$ and $g_{\alpha\beta} < 0$. If \tilde{A} denotes the transformation of TM uniquely defined by $g(\tilde{A}(u), v) = A(u, v)$ for all vectors u, v over the same base point, then saying A is dominantly non-negative is equivalent to requiring $-\tilde{A}(u)$ be future time-like or null whenever u is future time-like, which is the usual statement of the dominant energy condition for A . The *weak energy condition* simply requires that T is observer non-negative definite, whereas the *strong energy condition* requires that \hat{T} is observer non-negative definite. Thus, by the Einstein equation, the strong energy condition is equivalent to requiring that $Ric = 8\pi\hat{T}$ be observer non-negative definite. As it is expected that the pressures and stresses are smaller than the mass energy, it is reasonable that $c(T) \leq 0$, so the weak energy condition is probably weaker than the strong energy condition. Notice the strong energy condition is really completely geometric, as it says simply $Ric(u, u) \geq 0$ for every time-like vector u , and Ric can be determined by the connection without reference to the metric. We say that the energy-stress tensor T satisfies the *dominant energy condition* provided that T is dominantly non-negative. Now, $\tilde{H} = 2\tilde{T} - c(T)id_{TM}$. Thus, the dominant energy condition holds for T , if and only if $-\tilde{H}(u) - c(T)u$ is future time-like or null for any future time-like vector $u \in TM$. However, thinking of $Ric = 4\pi H$ with H the total energy, it would now seem reasonable to require that H and hence also Ric be dominantly non-negative.

8 THE COSMOLOGICAL CONSTANT

If we include the cosmological constant Λ in the Einstein equation, it becomes

$$Ric - (1/2)Rg + \Lambda g = 8\pi T, \quad (8.1)$$

which is of course the same as

$$Ric - (1/2)Rg = 8\pi[T - (1/8\pi)\Lambda g], \quad (8.2)$$

which means we view the equation here as having a modified energy-stress tensor

$$T_\Lambda = T - (1/8\pi)\Lambda g. \quad (8.3)$$

We then have $c(T_\Lambda) = c(T) - (1/2\pi)\Lambda$, so the effective energy-stress tensor of the gravitational field is

$$T_g = T - c(T)g - (1/2\pi)\Lambda g, \quad (8.4)$$

and the effective total energy-stress tensor serving as source is

$$H = 2T - c(T)g - (1/2\pi)\Lambda g. \quad (8.5)$$

In any case, as $div g = 0$, it follows that our conclusions about the energy-momentum flow of the gravitational field from (5.10) and (5.11) remain valid, even in the presence of a cosmological constant.

9 QUASI LOCAL MASS

The problem of defining the energy contained in a space-like hypersurface has led to many different definitions of the mass enclosed by a closed space-like surface contained in an arbitrary spacetime manifold, and these go by the general name quasi-local mass. Typically, they are defined by some kind of surface integral and give an indication of the mass enclosed by the space-like surface. For an extensive survey of these we refer the interested reader to [16]. In particular, the results of [13] on the Penrose quasi-local mass show that the results can be interesting when the space-like surface is not the boundary of a space-like hypersurface. A list of desirable properties of any definition of quasi-local mass is given in [8], where in particular it is shown that for their definition, the quasi-local mass enclosed by a space-like surface S is non-negative provided that the dominant energy condition holds and the surface S is the boundary of a hypersurface, Ω . It is further assumed that the boundary surface S has positive Gauss curvature and space-like mean curvature vector, and consists of finitely many connected components. The local energy condition assumed is framed in terms of the second fundamental form of the hypersurface, and in particular, we can see that for a geodesic hypersurface it reduces to the condition that the scalar curvature of the hypersurface, Ω , is non-negative, since in that case the second fundamental form vanishes (extrinsic curvature zero). But, the scalar curvature of the space-like hypersurface Ω is $2E(u, u) = 16\pi T(u, u)$, where u is a time-like future pointing unit normal field on Ω . So if the energy-stress tensor satisfies the weak energy condition in this case, then the energy density as seen by observers riding the hypersurface is non-negative, and we would simply integrate $(1/8\pi)E(u, u)$ over the hypersurface to find the energy inside, which is clearly non-negative. The amazing result in [8] is that the quasi-local mass defined there is defined in terms of integrals over the boundary S . For instance, their results show if the energy inside any one component of S vanishes, then S is connected and Ω is flat ([8], Theorem 1, page 183), and thus the result shows that the energy in Ω is in some sense determined by the geometry of the boundary and its mean curvature vector under the assumptions stated above. In order to make use of the total energy-stress tensor, H , in a similar setting, one would assume an appropriate energy condition, and then for a space-like hypersurface K with future time-like unit normal field u , it is natural to consider $H(u, u)\mu_K$ where μ_K is the volume form due to the Riemannian metric induced on K . The integral of $H(u, u)\mu_K$ over all of Ω should be the total energy inside K . More generally, if we assume that H is dominantly non-negative, that is, it satisfies the dominant energy condition, then given another reference future pointing time-like vector field k , one might then integrate $H(u, k)\mu_K$ over K . If a 2-form α can be found on K satisfying $d\alpha = H(u, k)\mu_K$, and if K is a 3-submanifold with boundary B , then by Stoke's theorem, the total energy inside K is the integral of α over the boundary B of K . In particular, we say that K is *instantaneously static* if there is an open set $U \subset M$ containing K and a vector field k on U which is future pointing and orthogonal to K and which satisfies Killing's equation, at each point of K . If $\omega = k^*$ is the dual 1-form to k , so $\omega(v) = g(k, v)$ for all vectors v , then this is equivalent to requiring $Sym(\nabla\omega)|_K = 0$ or equivalently that $(d\omega)|_K = 2\nabla\omega|_K$, which to be perfectly clear means that the difference $d\omega - 2\nabla\omega$ as calculated on U in fact is zero at each point of K . Then as in the Komar [7] integral (see [11], [17], pages 287-289 or [12], pages 149-151) it follows that

$$(-1/8\pi)d * d\omega = (1/4\pi)Ric(u, k) = H(u, k)\mu_K. \quad (9.1)$$

Here, $*$ denotes the Hodge star operator on M . Thus, $(-1/8\pi) * d\omega$ is a potential for the total energy on K . For any closed 2-submanifold S of K we define the quasi-local total energy $H(S, k)$

by

$$H(S, k) = -\frac{1}{8\pi} \int_S *d(k^*). \quad (9.2)$$

Thus, if $K_0 \subset K$ is a submanifold with boundary $S = \partial K_0$, then by Stoke's Theorem, (9.2) becomes

$$H(S, k) = -\frac{1}{8\pi} \int_{K_0} d * d(k^*) = \int_{K_0} H(u, k) \mu_K, \quad (9.3)$$

which is then non-negative if H is observer non-negative definite. Of course, this is the same as requiring Ric be observer non-negative definite, a purely geometric requirement. Thus, if $H(S, k) = 0$, with $S = \partial K_0$, then by (9.3), under the assumption that H is dominantly non-negative, we would conclude that $Ric(u, k) = 0$ on K_0 . But, this means that $Ric(u, u) = 0$ on K_0 , and this means that the scalar curvature of K_0 is identically zero. In particular, if K_0 has constant sectional curvature, this would imply that K_0 is actually flat. Notice that if we have an asymptotically flat spacetime with a global time-like killing vector field orthogonal to a spacelike slice, normalized to be a unit vector at spatial infinity, then our definition of the quasi-local total energy would be exactly the Komar mass which is well known in the literature [16]. Thus in the expression $H(S, k)$, the normalization for k is determined by requiring that it be of unit length at the event at which the observer is located. If the observer is located so that S is in the observer's causal past, then it would seem we must assume that the domain of k contains this past light cone. In general, if k is a Killing field on all of the open set U , then being orthogonal to K means ([17], page 119, (6.1.1)) that also $\omega \wedge d\omega = 0$, where $\omega = k^*$. Then (see [17], page 443, (C.3.12)) we find, using $f = \ln(|g(k, k)|)$,

$$d\omega = -\omega \wedge df, \quad (9.4)$$

and using the fact that here $*[\omega \wedge df] = -(e^{f/2} D_n f) \mu_S$, where n is the outward unit normal to $S = \partial K_0$, and $\mu_S = dA$ is the area 2-form on S , we obtain finally,

$$H(S, k) = -\frac{1}{8\pi} \int_S e^{f/2} D_n f dA. \quad (9.5)$$

In particular, for the vacuum Schwarzschild solution with mass M , taking the Killing field $k = \partial_t$, we see easily that the mass calculated using the integral (9.5) gives the value M for the mass enclosed by any sphere centered at the "origin" when we normalize the Killing field to be a unit vector at infinity. On the other hand, if we calculate that value of the integral by normalizing to make the Killing vector a unit at radial coordinate r_0 , as $H(S, k)$ is homogeneous in k , the normalizing constant comes out resulting in

$$M_{r_0} = \frac{M}{[1 - \frac{2M}{r_0}]^{1/2}}. \quad (9.6)$$

Keeping in mind this is now the total energy, gravitational and massive, this indicates a problem develops as $r_0 \rightarrow 2M$, even though we know it is not a real problem for the spacetime. This indicates the problem is probably due to the normalization involving the Schwarzschild radial coordinate which obviously breaks down at $r_0 = 2M$. After all, what we are integrating is equivalent by Stoke's Theorem to integrating $H(u, k) \mu_K$, when $S = \partial K_0$, and we really want to be integrating $H(u, u) \mu_K$. We do not have the actual potential.

10 APPENDIX ON ANALYTIC CONTINUATION

Suppose that $S : E \times E \rightarrow F$ is any symmetric bilinear map of vector spaces. Let the quadratic function f_S be defined by $f_S(x) = S(x, x)$. Then f_S determines S . This is just (what mathematicians would call) polar decomposition:

$$S(x, y) = (1/4)[f_S(x + y) - f_S(x - y)]. \quad (10.1)$$

That is, we note that

$$f_S(x \pm y) = f_S(x) + f_S(y) \pm 2S(x, y), \quad (10.2)$$

so subtracting the "minus" equation from the "plus" equation of (10.2) gives (10.1). In particular, if $f_S = 0$, then $S = 0$. But more is true. For, suppose E is a topological vector space and that f_S is constant on the open subset $U \subset E$. Choosing $x \in U$ and y sufficiently "small", we can assume that both $x + y$ and $x - y$ belong to U in which case we have $S(x, y) = 0$ from the polar decomposition (10.1). But from homogeneity, it follows that $S(x, y) = 0$, for every $x \in U$, and every $y \in E$. Since S is symmetric, it follows that $S(y, x) = 0$, for every $x \in U$, and every $y \in E$. Now, if $x, y \in E$ are any vectors, simply choose a vector $x_0 \in U$, and we have $S(x, x_0) = 0$, so $S(x, y) = S(x, y - x_0)$, and we will see that $S(x, y - x_0) = 0$. For there is a small scalar $t \neq 0$ so that $x_0 + t(y - x_0)$ belongs to U , and therefore, $0 = S(x, x_0 + t(y - x_0)) = S(x, x_0) + tS(x, y - x_0)$. Since $S(x, x_0) = 0$, it now follows that $S(x, y - x_0) = 0$, and thus finally we have $S(x, y) = 0$. The argument can be simplified by using differentiation (see for instance [15], page 72), but we prefer to give a purely algebraic argument here. Using differentiation, we next generalize easily to n -linear maps, but a telescoping algebraic argument could be applied with a little more effort.

Thus, if S is a continuous symmetric n -linear map of the Banach space E into the Banach space F , and if we define the monomial function $f_S : E \rightarrow F$ by the rule $f_S(x) = S(x, x, x, \dots, x) = Sx^{(n)}$, then f_S is an analytic function. In fact, if $x_1, x_2, x_3, \dots, x_n \in E$, then differentiating we find

$$D_{x_1} D_{x_2} D_{x_3} \dots D_{x_n} f(a) = (n!) S(x_1, x_2, x_3, \dots, x_n), \quad a \in E. \quad (10.3)$$

From (10.3), we see very generally that if U is any open subset of E on which f_S is constant, then in fact, $S = 0$, since we can choose $a \in U$. Indeed, if $a \in U$, since f_S is constant on U , it follows that the derivative on the left side of the equation (10.3) is 0, and hence also the right side, for every possible choice of vectors $x_1, x_2, x_3, \dots, x_n \in E$. But notice that a does not appear on the right hand side of (10.3), only $S(x_1, x_2, x_3, \dots, x_n)$, and the vectors $x_1, x_2, x_3, \dots, x_n$ can be chosen arbitrarily. Thus, $S = 0$ follows. This is just a very special case of the principle of analytic continuation. However, to see (10.3), it served our purpose for the Einstein equation to only examine the case where $n = 2$ and the vector spaces are finite dimensional, so the vector topologies are unique and any bilinear map is therefore continuous and in fact smooth. In this case, we have for any $x, w \in E$, the easily checked fact that

$$D_w f_S(x) = S(x, w) + S(w, x) = 2S(x, w). \quad (10.4)$$

Differentiating again, since as a function of x alone $D_w f_S$ is obviously linear, it follows that for any $x, v, w \in E$,

$$D_v D_w f_S(x) = 2S(v, w). \quad (10.5)$$

It should be obvious how the general case of (10.3) from the method of showing (10.5). In any case, clearly, if f_S is constant on some (no matter how small) non-empty open subset U of E , then choosing $x \in U$ would give $D_v D_w f_S(x) = 0$ for every $v, w \in E$ and every $x \in U$. Thus the right hand side of (10.5) vanishes for every $v, w \in E$ and this says $S = 0$. In our application, we are taking the vector space to be the tangent space to M at a specific point, say $m \in M$, so $E = T_m M$. Then, either half of the light cone, future or past, is an open subset of $T_m M$ and therefore, if S is a symmetric second rank tensor at m whose quadratic form vanishes on the future light cone or vanishes on the past light cone, or on any small open subset of the light cone, then $S = 0$. It follows that in general relativity theory, symmetric tensors at a point can be specified by their monomial forms on the future or on the past light cone at that point, and hence it suffices, by homogeneity of the monomial forms to limit consideration to time-like unit vectors. We might say in fact that in general relativity, any tensor equation should be viewed as a tensor valued symmetric tensor, and on all slots for which there is symmetry one need only evaluate by using the same arbitrary unit time-like vector in each of those slots. For instance, if $A(a, b, c, d, e)$ is a 5 – linear transformation which is symmetric in the first three slots, then considering $A(a, b, c, -, -)$ as a third rank symmetric tensor which is second rank tensor valued, it is completely determined by knowing $A(u, u, u, -, -)$ for every time-like unit vector $u \in TM$. This is what we call the symmetric tensor observer principle. In some sense, this is the principle of relativity. If we apply this to the electromagnetic field, F , for instance, as it is anti-symmetric, that is, a 2-form, it should be viewed as a 1-form valued 1-form. Thus, it is determined by giving the 1-form $F(u)$ for each time-like unit vector u . In fact, we can say that an observer with velocity u holding a test charge Q should feel the force $QF(u)$, so $F(u)$ is the force 1-form per unit charge experienced by an observer holding an electric charge. Since the force vector in the instantaneous rest frame is orthogonal to the velocity, we must have $F(u, u) = 0$ here. As this experiment could be carried out by any observer, this means that $F(u, u) = 0$ for every time-like unit vector, so $Sym(F) = 0$, by the symmetric tensor observer principle, and therefore $F = Alt(F)$ must be a 2-form. We can similarly say that any force field which applies to certain objects called "charges", according to a scalar measure of that charge making force felt by any observer holding the charge exactly proportional to the charge, would be described by a 2-form.

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