

GROWING WORDS IN THE FREE GROUP ON TWO GENERATORS

BOBBE J. COOPER AND ERIC S. ROWLAND

ABSTRACT. This paper is concerned with minimal length representatives of equivalence classes of F_2 under $\text{Aut } F_2$. We give a simple inequality characterizing words of minimal length in their equivalence class. We consider an operation that “grows” words from other words, increasing the length, and we study *root words* — minimal words that cannot be grown from other words. Root words are “as minimal as possible” in the sense that their characterization is the boundary case of the minimality inequality. The property of being a root word is respected by equivalence classes, and the length of each root word is divisible by 4.

1. INTRODUCTION

The free group with n generators F_n is quite large, so when working with elements of F_n it is desirable to work instead with equivalence classes of elements as much as possible. In this paper we consider equivalence classes of F_n under the full automorphism group. One would like to find minimal-length representatives of each equivalence class. In 1936 J. H. C. Whitehead [6, 7] made this goal practical by exhibiting a finite set of *Whitehead automorphisms* with the property that if two words w and v in F_n are equivalent under an automorphism of F_n and v is of minimal length in its equivalence class, then v is the image of w under a sequence of Whitehead automorphisms. Furthermore, the words obtained after applying each automorphism in this sequence are strictly decreasing in length until the minimal length is attained, after which the automorphisms leave the length fixed.

Because there are only finitely many Whitehead automorphisms, the theorem allows one to determine whether any two words in F_n are equivalent, by a finite calculation. Moreover, by individually applying all the Whitehead automorphisms to a word, it allows one to determine whether the word is of minimal length in its equivalence class by a finite calculation.

In Section 2 we define our notation and state Whitehead’s theorem. Section 3 uses subword counts to establish a simple characterization of minimal words in F_2 . In Section 4 we define root words and establish some of their properties. We consider an operation that “grows” words from other words, increasing the length. Root words are “as minimal as possible” in the sense that their characterization is the boundary case of the minimality inequality. We prove that the property of being a root word is respected by equivalence classes, and the length of each root word is divisible by 4.

Date: September 16, 2009.

We thank Dennis Garity for comments and suggestions throughout the development of this paper.

This work was supported in part by NSF grant DMS 0139678.

2. PRELIMINARIES

We start with some notation and definitions. F_n denotes the free group on n generators. For $F_2 = F(a, b)$ the generators are a and b , and we will denote the inverse of a generator x by \bar{x} . The set of letters — the generators and their inverses — is referred to as L_n . Thus $L_2 = \{a, b, \bar{a}, \bar{b}\}$. We will use x and y as variables over L_2 where usually $x \notin \{y, \bar{y}\}$.

A *word* is an element $w \in F_n$. The identity word is represented by 1. The relation \sim is defined on words in F_n by $w \sim v$ if $S(w) = v$ for some automorphism $S \in \text{Aut } F_n$; in this case we say that w and v are *equivalent*. $|w|$ denotes the length of the word w (after any adjacent inverses are canceled).

Denote the inner automorphism group of F_n by $I = \text{Inn } F_n$. Define another equivalence relation \sim_I on words in F_n by $w \sim_I v$ if $S(w) = v$ for some automorphism $S \in I$. We denote by $[w]$ the equivalence class of w under this relation. For example, if $w = ab\bar{a}\bar{b}$, then $[w]$ contains the words $ab\bar{a}\bar{b}$, $\bar{b}\bar{a}b\bar{a}$, $\bar{a}\bar{b}ab$, and $\bar{b}a\bar{b}a$, as well as all words such as $\bar{a}\bar{b}ab\bar{a}b\bar{b}a$ in which inverses can be canceled from the beginning and end of the word to produce one of these four.

The main goal of this paper is to study minimal length representatives of equivalence classes under \sim . We do this by examining representatives of equivalence classes under \sim_I , as this simplifies computations by allowing us to consider the initial letter of a word to be adjacent to the final letter. From here on, we abuse notation by suppressing the brackets around $[w]$ and write w for the equivalence class under \sim_I of w . For $S \in \text{Aut } F_n$, note that $S(w)$ is well-defined as an equivalence class under \sim_I because I is a normal subgroup of $\text{Aut } F_n$.

Define the length $|w|$ of an equivalence class w under \sim_I to be the minimal length of a word in the equivalence class. It may be that not all words in the equivalence class have the same length because we can cancel inverses from the beginning and end of a word.

Now, we set up our notation to discuss Whitehead's theorem.

A *Type I automorphism* is an automorphism $S \in \text{Aut } F_n$ which permutes L_n . Note that a permutation S of L_n can be extended to a Type I automorphism if $S(\bar{y}) = \overline{S(y)}$ for all $y \in L_n$. We will refer to Type I automorphisms as *permutations*.

Let $x \in L_n$ and $A \subset L_n$. Define a map $S : L_n \rightarrow F_n$ by

$$S(y) = \bar{x}^{[y \in A]} y x^{[y \in A]},$$

where $[p]$ is 1 if p is true and 0 if p is false. Since $S(y)^{-1} = S(\bar{y})$, this map extends to an automorphism $S \in \text{Aut } F_2$. We write $S = (A, x)$ and call S a *Type II automorphism*. Generally, we take $x, \bar{x} \notin A$.

A *one-letter automorphism* is a Type II automorphism $S = (A, x)$ where the set A contains only one element.

Example. The one-letter automorphism $(\{a\}, b)$ maps $a \mapsto ab$ and $\bar{a} \mapsto \bar{b}\bar{a}$ and leaves b, \bar{b} fixed.

Definition. A word $w \in F_n$ is minimal if $|w| \leq |S(w)|$ for all $S \in \text{Aut } F_n$.

Example. Let $S = (\{a\}, b) \in \text{Aut } F_2$. We have $S(\bar{b}\bar{a}) = \bar{b}\bar{b}\bar{a} = \bar{a}$. Therefore the word $\bar{b}\bar{a}$ is not minimal.

We now give Whitehead's theorem.

Theorem (Whitehead). *If $w, v \in F_n$ such that $w \sim v$ and v is minimal, then there exists a sequence S_1, S_2, \dots, S_m of Type I and Type II automorphisms such that $S_m \cdots S_2 S_1(w) = v$ and for $1 \leq k \leq m$, $|S_k S_{k-1} \cdots S_2 S_1(w)| \leq |S_{k-1} \cdots S_2 S_1(w)|$, with strict inequality unless $S_{k-1} \cdots S_2 S_1(w)$ is minimal.*

Whitehead's proof of the theorem was topological. Rapaport [3] provided an algebraic proof, which was simplified by Higgins and Lyndon [2].

We let $n = 2$ for the remainder of the paper.

3. MINIMALITY

We begin the discussion of minimal words by describing two useful types of (contiguous) subwords — two-letter subwords and maximal subwords of identical letters.

The two-letter subwords of $w = aa\bar{a}\bar{b}\bar{b}\bar{a}b\bar{a}\bar{b}$ are $aa, a\bar{a}, \bar{a}\bar{b}, \bar{b}\bar{b}, \bar{b}\bar{a}, \bar{a}b, b\bar{a}, \bar{a}\bar{b}$, and $\bar{b}a$. There are n two-letter subwords (counted with multiplicity) in a length- n word, since one subword is composed of the final letter and the initial letter. (In particular, the set of two-letter subwords of a word does not depend on the (minimal length) representative of its equivalence class under \sim_I .)

The two-letter subwords of a word record adjacencies. When considering the effect of an automorphism on a word, it suffices to track additions and cancellations *between* the two letters of each two-letter subword. For this reason the two-letter subwords are useful in characterizing minimality.

Let $\{v\}_w$ denote the number of occurrences of the substring v in w , and let $(v)_w = \{v\}_w + \{v^{-1}\}_w$ denote the total number of occurrences of v and v^{-1} in w .

Example. *Let $w = aa\bar{a}\bar{b}\bar{b}\bar{a}b\bar{a}\bar{b}$. The two-letter subword counts are $(aa)_w = 2$, $(ab)_w = 1$, $(\bar{a}\bar{b})_w = 2$, $(ba)_w = 1$, $(bb)_w = 1$, and $(\bar{a}b)_w = 2$.*

An x -string is a subword of a word w of the form x^n that is not a substring of x^{n+1} in w . Let $\lambda(w)$ denote the length of the longest x -string in w . For example, $\lambda(aa\bar{a}\bar{b}\bar{b}\bar{a}b\bar{a}\bar{b}) = 3$. A word w is *alternating* if $\lambda(w) = 1$.

By observing that each a -string and \bar{a} -string has as its neighbors b or \bar{b} and that each b -string and \bar{b} -string likewise has as its neighbors a or \bar{a} , we prove the following theorem.

Theorem 1. *If $w \in F_2$ and $x, y \in L_2$, then $(xy)_w = (yx)_w$.*

Proof. We may assume $x \notin \{y, \bar{y}\}$ since otherwise the theorem is trivial. Each occurrence of the subwords xy and $\bar{x}y$ in w must be followed by either yx or $y\bar{x}$ (with possibly an intermediate y -string). Similarly each occurrence of the subwords $y\bar{x}$ and $\bar{y}x$ in w must be followed by either $\bar{x}y$ or $\bar{x}\bar{y}$ (with possibly an intermediate \bar{x} -string). Therefore

$$\begin{aligned} \{xy\}_w + \{\bar{x}y\}_w &= \{yx\}_w + \{y\bar{x}\}_w, \\ \{y\bar{x}\}_w + \{\bar{y}x\}_w &= \{\bar{x}y\}_w + \{\bar{x}\bar{y}\}_w. \end{aligned}$$

By definition, $\{xy\}_w + \{\bar{y}\bar{x}\}_w = (xy)_w$ and $\{yx\}_w + \{\bar{x}\bar{y}\}_w = (yx)_w$. We add the above equations and use these relations to obtain $(xy)_w = (yx)_w$. \square

Note that the Type II automorphism $(\{y, \bar{y}\}, x)$ with $x \notin \{y, \bar{y}\}$ is an inner automorphism on F_2 , since it conjugates x by y and also (trivially) conjugates y by y , and x, y generate F_2 . Consequently, automorphisms of this form do not change

the length of a word, and thus by Whitehead's Theorem it suffices to consider one-letter automorphisms when determining the minimality of a word in F_2 . This allows us to prove the following, a special case of a theorem of Rapaport [3, Theorem 7].

Lemma 2. *w is minimal if and only if for all $x, y \in L_2$ with $x \notin \{y, \bar{y}\}$ we have $(y\bar{x})_w \leq (yx)_w + (yy)_w$.*

Proof. The automorphism $(\{y\}, x)$ causes cancellations in w only in the subwords $y\bar{x}$ and $x\bar{y}$. The total number of cancellations is therefore $(y\bar{x})_w$. Similarly $(\{y\}, x)$ causes additions to w in the subwords yx , $\bar{x}\bar{y}$, yy , and $\bar{y}\bar{y}$, totaling $(yx)_w + (yy)_w$. Thus, the left side of the inequality is the number of cancellations in w under the automorphism $(\{y\}, x)$, while the right side is the number of additions under that automorphism. w is minimal if and only if the inequality holds for each choice of x, y . \square

There are eight distinct one-letter automorphisms. Each of these can be written as the product of a cycle and another one-letter automorphism: $(\{y\}, x) = (\{y, \bar{y}\}, x)(\{\bar{y}\}, \bar{x})$, so there are at most four distinct one-letter automorphisms after reducing mod I . Therefore we have the following.

Lemma 3. *To determine the minimality of a word, it suffices to consider the four one-letter automorphisms $(\{a\}, b)$, $(\{a\}, \bar{b})$, $(\{b\}, a)$, and $(\{b\}, \bar{a})$, or more generally, given x, y , the four one-letter automorphisms $(\{y\}, x)$, $(\{y\}, \bar{x})$, $(\{x\}, y)$, and $(\{x\}, \bar{y})$.*

The previous lemma provides the following characterization of minimal words. It is a substantially simpler test than that given by Sanchez [4].

Theorem 4. *$w \in F_2$ is minimal if and only if $|(ab)_w - (a\bar{b})_w| \leq \min((aa)_w, (bb)_w)$, or more generally, for any x, y , $|(yx)_w - (y\bar{x})_w| \leq \min((xx)_w, (yy)_w)$.*

Proof. There are eight inequalities to be considered in Lemma 2 over L_2 ; applying Lemma 3 to these reduces the number of distinct inequalities to four, giving that w is minimal if and only if $(ab)_w \leq (a\bar{b})_w + (aa)_w$, $(a\bar{b})_w \leq (ab)_w + (aa)_w$, $(ba)_w \leq (b\bar{a})_w + (bb)_w$, and $(b\bar{a})_w \leq (ba)_w + (bb)_w$. These four are equivalent to $|(ab)_w - (a\bar{b})_w| \leq (aa)_w$ and $|(ba)_w - (b\bar{a})_w| \leq (bb)_w$, and the result follows from Theorem 1. The same argument goes through with a, b replaced by any x, y . \square

Corollary 5. *If w is a minimal word with no \bar{a} or \bar{b} , then $(ab)_w \leq \min((aa)_w, (bb)_w)$.*

Corollary 6. *If w and v are minimal words, and we choose representatives of w, v mod \sim_I with the same initial letter, then wv is minimal.*

Proof. The inequality of Theorem 4 holds for w and for v , so it holds for wv by addition because the two-letter subwords are preserved. \square

4. ROOT WORDS

A result of Corollary 6 was observed by Virnig [5], namely that if w is minimal and we choose a representative of w mod \sim_I which begins with a , then aw is also minimal. This motivates the following definition.

Definition. *A word v is a child of a word w if $v = xw$ for some $x \in L_2$, where either w is the empty word or we choose a representative of w mod \sim_I which begins with x .*

A child is simply a word obtained by increasing the length of an x -string in another word. As we consider children, we also have parents, although a word does not necessarily have a unique parent. For example, the parents of $aabb\bar{a}\bar{a}bb$ are $abb\bar{a}\bar{a}bb$, $aab\bar{a}\bar{a}bb$, $aabb\bar{a}\bar{b}$, and $aabb\bar{a}\bar{a}b$. A child of a minimal word is necessarily minimal by Corollary 6.

We are interested in characterizing words which cannot be grown as children of other words. These will be the essentially new minimal words.

Definition. A root word is a minimal word that is not a child of any minimal word.

Example. The words $ab\bar{a}\bar{b}$ and $ab\bar{a}\bar{b}$ are root words; they are minimal by Theorem 4 and are not children of any minimal word since in both cases $\lambda(w) = 1$. The word $aabb$ is also a root word, since it is minimal and its parents abb and aab are not.

Root words are the words for which Theorem 4 holds for equality.

Theorem 7. $w \in F_2$ is a root word if and only if $|(ab)_w - (a\bar{b})_w| = (aa)_w = (bb)_w$, or more generally for any x, y , $|(yx)_w - (y\bar{x})_w| = (yy)_w = (xx)_w$.

Proof. By Theorem 4, we have $|(ab)_w - (a\bar{b})_w| \leq (aa)_w$ and $|(ab)_w - (a\bar{b})_w| \leq (bb)_w$. A minimal word w is a root word if and only if shortening any x -string of length ≥ 2 in w causes the word to lose minimality. Shortening such an x -string corresponds to decrementing $(aa)_w$ or $(bb)_w$ (or more generally $(yy)_w$ or $(xx)_w$), so therefore w is a root word precisely when both inequalities hold for equality. \square

From this characterization we can derive many properties of root words.

Corollary 8. If w is a root word, then $(a)_w = (b)_w$.

Proof. The only two-letter subwords in w with unequal generator weights are aa , $\bar{a}\bar{a}$, bb , and $\bar{b}\bar{b}$, but by the previous theorem $(aa)_w = (bb)_w$. \square

Corollary 9. Let $n \geq 1$. Then w is a root word if and only if w^n is a root word.

Proof. Multiplying w by itself preserves equality in Theorem 7. Likewise, taking the n th root of w^n preserves equality. \square

Theorem 10. If w is a root word, then $|w|$ is divisible by 4.

Proof. $|w| = (aa)_w + (bb)_w + (ab)_w + (ba)_w + (a\bar{b})_w + (\bar{b}a)_w$ because these subwords and their inverses constitute the set of two-letter subwords. By Theorem 1 and Theorem 7 this simplifies to

$$\begin{aligned} |w| &= 2(aa)_w + 2(ab)_w + 2(a\bar{b})_w \\ &= 2|(ab)_w - (a\bar{b})_w| + 2(ab)_w + 2(a\bar{b})_w. \end{aligned}$$

If $(ab)_w \geq (a\bar{b})_w$ then $|w| = 4(ab)_w$, and if $(ab)_w < (a\bar{b})_w$ then $|w| = 4(a\bar{b})_w$. In either case $|w|$ is divisible by 4. \square

The following result gives an upper bound on the length of x -strings in root words.

Theorem 11. If w is a root word, then $\lambda(w) \leq \frac{|w|}{4} + 1$.

Proof. We have $(ab)_w + (a\bar{b})_w \geq |(ab)_w - (a\bar{b})_w| = (aa)_w$. From the proof of Theorem 10 we therefore obtain $|w| = 2((aa)_w + (ab)_w + (a\bar{b})_w) \geq 4(aa)_w$, so $\lambda(w) \leq \max((aa)_w, (bb)_w) + 1 = (aa)_w + 1 \leq \frac{|w|}{4} + 1$. \square

Furthermore, there exists a root word of length $4n$ that achieves $\lambda(w) = \frac{|w|}{4} + 1$, namely $a^{n+1}(ba)^{n-1}b^{n+1}$, which can be shown to be a root word by Theorem 7.

We will show that all minimal words which are equivalent to a root word under any automorphism are also root words. For this, we need a series of lemmas.

The following lemma describes the effect of a one-letter automorphism on a word.

Lemma 12. *Let w be a word, and let $S = (\{y\}, x)$ with $x \notin \{y, \bar{y}\}$. Let $v = S(w)$; then*

$$\begin{aligned} (yy)_v &= (y\bar{x}y)_w \\ (yx)_v &= (yx)_w + (yy)_w \\ (y\bar{x})_v &= (y\bar{x})_w - (y\bar{x}y)_w \\ (xx)_v &= (yx)_w - (yx\bar{y})_w + (xx)_w - (y\bar{x}\bar{x})_w. \end{aligned}$$

Also, $(yy)_v - (xx)_v = (y\bar{x})_w - (yx)_w - (xx)_w$.

Proof. The subword $(yy)_w$ only appears in v as a result of cancellations in $(y\bar{x}y)_w$ in w , which yields the first equality. The subword $(yx)_w$ remains fixed under S , and also arises in v from $(yy)_w$ in w , which yields the second equality. Next, $(y\bar{x})_w$ remains fixed under S , unless it appears in $(y\bar{x}y)_w$, which yields the third equality. Next, $(xx)_w$ remains fixed under S unless it appears in $(y\bar{x}\bar{x})_w$, and also arises in v from $(yx)_w$, unless this $(yx)_w$ appears in $(yx\bar{y})_w$ in w , which yields the fourth equality. For the final equality, subtract the first two equalities and use the facts that $(yx\bar{y})_w = (y\bar{x}\bar{y})_w$ (since $(yx\bar{y})^{-1} = y\bar{x}\bar{y}$) and that $(y\bar{x})_w = (y\bar{x}y)_w + (y\bar{x}\bar{y})_w + (y\bar{x}\bar{x})_w$. \square

Definition. *An automorphism S is level on a word w if $|S(w)| = |w|$.*

For example, all Type I automorphisms are level on all words. The following result characterizes level one-letter automorphisms.

Lemma 13. *If $x \notin \{y, \bar{y}\}$, then $(\{y\}, x)$ is a level automorphism on w if and only if $(y\bar{x})_w = (yx)_w + (yy)_w$.*

Proof. As in the proof of Lemma 2, the automorphism $(\{y\}, x)$ causes cancellations in w only in the subwords $y\bar{x}$ and $x\bar{y}$. The total number of cancellations is therefore $(y\bar{x})_w$. Similarly, $(\{y\}, x)$ causes additions to w in the subwords yx , $\bar{x}y$, yy , and $\bar{y}y$, totaling $(yx)_w + (yy)_w$. We have that $(\{y\}, x)$ is a level automorphism on w if and only if the number of additions it causes is equal to the number of cancellations it causes, so $(y\bar{x})_w = (yx)_w + (yy)_w$. \square

Lemma 14. *If w is a root word, and S is a one-letter automorphism which is level on w , then $S(w)$ is a root word.*

Proof. Let $v = S(w)$, and $S = (\{y\}, x)$. By Lemma 12, $(yy)_v - (xx)_v = (y\bar{x})_w - (yx)_w - (xx)_w$. Since w is a root word, $(yy)_w = (xx)_w$ from Theorem 7; and $(y\bar{x})_w = (yx)_w + (yy)_w$ from Lemma 13, so $(y\bar{x})_w - (yx)_w - (xx)_w = 0$, which means that $(yy)_v = (xx)_v$. By Lemma 12, $|(yx)_v - (y\bar{x})_v| = |(yx)_w + (yy)_w - (y\bar{x})_w - (y\bar{x}y)_w| = |-(y\bar{x}y)_w| = (y\bar{x}y)_w = (yy)_v$. Therefore, by Theorem 7, v is a root word. \square

Theorem 15. *If w is a root word, $w \sim v$, and $|w| = |v|$, then v is a root word.*

Proof. Note that the conditions given in Theorem 7 are symmetric in permutations of the generators, by comparison with Lemma 1. This means that permutations preserve the property of being a root word. By the lemma above, root words remain root words under level one-letter automorphisms. By Whitehead's Theorem, each $w \sim v$ is connected to w by a chain of one-letter automorphisms, cycles, and permutations. Therefore each v is a root word. \square

In light of this theorem we refer to an equivalence class containing a root word as a *root word equivalence class*, since all minimal words in it are root words. In a forthcoming paper [1] we determine the structure of root word equivalence classes and compute all root words of length ≤ 20 . This data is available from the second author's web site.

We conclude by showing that alternating root words are characterized by being level under the one-letter automorphisms.

Theorem 16. *The following are equivalent:*

- (1) *w is a minimal alternating word.*
- (2) *w is an alternating root word.*
- (3) *All four distinct one-letter automorphisms are level on w .*

Proof. Assume (1). If w is a minimal alternating word, then $(aa)_w = (bb)_w = 0$, so $(ab)_w = (a\bar{b})_w$ by Theorem 4. Therefore, by Theorem 7, w is a root word, so we have (2).

Assume (2) and let $S = (\{x\}, y)$. Because $(aa)_w = (bb)_w = 0$ and $(ab)_w = (a\bar{b})_w$, we have $(xy)_w = (x\bar{y})_w$ and $(xx)_w = 0$ for all $x, y \in L_2$, $x \notin \{y, \bar{y}\}$. Therefore the number of cancellations caused by S is equal to the number of additions, and the length of w does not change. Thus we have (3).

Let all one-letter automorphisms be level on w in accordance with (3). This implies $(ab)_w - (a\bar{b})_w = (aa)_w$, $(a\bar{b})_w - (ab)_w = (aa)_w$, $(ab)_w - (a\bar{b})_w = (bb)_w$, and $(a\bar{b})_w - (ab)_w = (bb)_w$ by Lemma 13, so $(aa)_w = (bb)_w = 0$ and $(ab)_w = (a\bar{b})_w$. Therefore w is minimal and alternating, so (1) holds. \square

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SCHOOL OF MATHEMATICS, 206 CHURCH ST. S.E., MINNEAPOLIS, MN 55455, USA

MATHEMATICS DEPARTMENT, TULANE UNIVERSITY, NEW ORLEANS, LA 70118, USA