Derivatives: Chain Rule, Implicit Differentiation, Logs

Section 3.4

Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product:

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Common forms of the Chain Rule:

- Can combine chain rule with product rule, quotient rule,...
- Power Rule combined with Chain Rule: if n is any real number and $u = g(x)$ is differentiable, then:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

or

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

- Exponential Function with base $a > 0$ combined with Chain Rule:

$$\frac{d}{dx}(a^x) = a^x lna$$

Proof:

Examples: Evaluate the derivatives of the following using the Chain Rule

1. $y = (x^3 - 1)^{100}$

2. $y = (x^3 - 1)^{100} \cdot (2x + 1)^2$

3. $y = e^{sec 3\theta}$

Section3.5

An **implicit function** is defined implicitly by a relation between x and y. (It is not of the usual form $y = f(x)$.) For example, the following are implicit functions:

$$x^2 + y^2 = 25$$
$$x^3 + y^3 = 6xy$$

For an implicit function, it is not necessary to solve the equation for y in terms of x to find the derivative of y. **Implicit differentiation** consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for $y' = \frac{dy}{dx}$.

Examples: Find $\frac{dy}{dx}$ using implicit differentiation

1. $x^3 + y^3 = 6xy$

2. $sin(x + y) = y^2 cos(x)$

Derivatives of Inverse Trig Functions

Definition of arcsine (inverse sine) function:

$$y = sin^{-1}x \qquad \text{means} \qquad siny = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Differentiating $siny = x$ with respect to x:

Derivatives of inverse trig functions

$$\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(cot^{-1}x) = -\frac{1}{1+x^2}$$

Examples: Find $\frac{dy}{dx}$

1. $y = \frac{1}{sin^{-1}x}$

2. $y = x\arctan(\sqrt{x})$

Section 3.6

Use implicit differentiation to find the derivatives of the logarithmic functions.

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

Proof:

Combining with the chain rule:

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{OR} \quad \frac{d}{dx}[\ln(g(x))] = \frac{g'(x)}{g(x)}$$

Examples:

1. Find $\frac{d}{dx}ln(sinx)$

2. Find $\frac{d}{dx}\sqrt{lnx}$

3. Differentiate the following by first taking logarithms of both sides, simplifying, and then differentiating implicitly (Logarithmic Differentiation)

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

Logarithmic Differentiation Steps

1. Take ln of both sides of the equation and use Log Properties to simplify
2. Differentiate implicitly with respect to x
3. Solve the resulting equation for y'

Using Logarithmic Differentiation, we can:

- Prove the Power Rule. For $f(x) = x^n$, then $f'(x) = nx^{n-1}$
  Proof:

- Differentiate functions of the form $\frac{d}{dx}[f(x)]^{g(x)}$
  Example: Differentiate $y = x^{\sqrt{x}}$
  Sln:

Derivative Rules for exponents and bases:

1. $\frac{d}{dx}(a^b) = 0$ where a and b are constants

2. $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$

3. $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(lna)g'(x)$

The number e as a limit

If $f(x) = ln(x)$, then $f'(x) = 1/x$, therefore $f'(1) = 1$.
From the definition of a derivative as a limit, we have: