CONFORTI-CORNUÉJOLS CONJECTURE VIA COMMUTATIVE ALGEBRA

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Abstract. Using the correspondence between clutters and square-free monomial ideals, we investigate an algebraic approach to a conjecture of Conforti and Cornuéjols which states that a clutter has the max-flow min-cut property if and only if it has the packing property. If a "minimal" counterexample $\mathcal{C}$ to this conjecture existed then powers of the corresponding monomial ideal I must have embedded primes. We show that the least power t such that I^t has embedded primes is exactly $t = \beta_1(\mathcal{C}) + 1$, where $\beta_1(\mathcal{C})$ is the matching number of $\mathcal{C}$. Our results also show that if existed, such a clutter $\mathcal{C}$ cannot be unmixed.

1. Introduction

There is a combinatorial realization of a square-free monomial ideal that can be manifested in a variety of ways, depending on the reader's background. Since the primary motivation of this paper is a conjecture from combinatorial optimization, we will use clutter language to state our problem and results. A *clutter* $\mathcal{C}$ consists of a finite set of points $V(\mathcal{C}) = \{x_1, \ldots, x_d\}$ and a family $E(\mathcal{C})$ of nonempty subsets of $V(\mathcal{C})$ with no non-trivial containments among them (i.e., if E_1 and E_2 are distinct elements in $E(\mathcal{C})$ then $E_1 \not\subseteq E_2$). The elements of $V(\mathcal{C})$ are called the *vertices* and the elements of $E(\mathcal{C})$ are called the *edges* of $\mathcal{C}$. Clutters are also known as *Sperner families* or *simple hypergraphs.*

Let k be a field. By identifying the points in $V(\mathcal{C})$ with the variables in a polynomial ring $R = k[x_1, \ldots, x_d]$, the natural one-to-one correspondence between square-free monomial ideals in R and clutters on $\{x_1, \ldots, x_n\}$ is given by

$$\mathcal{C} \leftrightarrow I(\mathcal{C}) = \langle x^E = \prod_{x \in E} x \mid E \in E(\mathcal{C}) \rangle.$$

The ideal $I(\mathcal{C})$ is referred to as the *edge ideal* of $\mathcal{C}$. This is the same as the construction of edge ideals of hypergraphs (cf. [12, 6]).

In [2], Conforti and Cornuéjols made the following conjecture (see Section 2 for the relevant definitions).

Conjecture 1.1 (Conforti-Cornuéjols)**.** A clutter $\mathcal{C}$ has the *max-flow min-cut* (MFMC) property if and only if $\mathcal{C}$ has the *packing* property.

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In 2001, Cornuéjols included Conjecture 1.1 as one of 18 conjectures for which he offered a prize (see [3, Conjecture 1.6]). It is well known (see also Section 2) that the MFMC property implies the packing property, so the point of Conjecture 1.1 is the other implication. In [5, Corollary 3.14] and [7, Corollary 1.6], it was shown that $\mathcal{C}$ satisfies MFMC if and only if the corresponding edge ideal $I(\mathcal{C})$ is *normally torsion-free*, that is, $I(\mathcal{C})^t = I(\mathcal{C})^{(t)}$ for all $t \geq 1$. This allows Conjecture 1.1 to be restated (cf. [4, Conjecture 4.18]) as: if $\mathcal{C}$ has the packing property, then $I(\mathcal{C})^t = I(\mathcal{C})^{(t)}$ for all $t \geq 1$ (or equivalently, $I(\mathcal{C})^t$ has no embedded primes for all $t \geq 1$).

If $\mathcal{C}$ is a graph in the classical sense, i.e., $I(\mathcal{C})$ is generated in degree two, then the packing property is equivalent to $\mathcal{C}$ being bipartite. In [11] it was shown that a graph is bipartite if and only if its edge ideal is normally torsion-free. Thus, Conjecture 1.1 has been verified for the case of a graph.

The goal of this paper is to examine algebraic properties of $I(\mathcal{C})$ when $\mathcal{C}$ is a "minimal" possible counterexample (if existed) to Conjecture 1.1. Our work can be viewed as the first step toward obtaining an algebraic solution to Conjecture 1.1. If such a $\mathcal{C}$ existed then $I(\mathcal{C})$ was not normally torsion-free, i.e., there existed a power $I(\mathcal{C})^t$ that has embedded primes. Thus, our focus is in investigating associated and embedded primes of powers of a square-free monomial ideals. More precisely, we study the associated and embedded primes of $I(\mathcal{C})^t$ in the case when every *minor* (see Section 2 for the definition) of $I(\mathcal{C})$ is normally torsion-free.

Our first main result, Theorem 3.9, shows that if, in addition to satisfying packing property, $\mathcal{C}$ is *unmixed*, then $I(\mathcal{C})$ is normally torsion-free. As a consequence (Corollary 3.10), a minimal counterexample, if existed, to Conjecture 1.1 cannot be an unmixed clutter. Our next results (Corollary 3.6 and Theorem 4.6) give a sharp lower bound of the power t for which $I(\mathcal{C})^t$ has embedded primes when $I(\mathcal{C})$ is not normally torsion-free. We show that in this case, $I(\mathcal{C})^t$ has no embedded primes for all $t \leq \beta_1(\mathcal{C})$, and $I(\mathcal{C})^{\beta_1(\mathcal{C})+1}$ must have embedded primes, where $\beta_1(\mathcal{C})$ is the *matching number* of $\mathcal{C}$.

Our method in proving Theorem 3.9 and Corollary 3.6 is to use induction on the power $I(\mathcal{C})^t$ of $I(\mathcal{C})$ and the number of vertices in $\mathcal{C}$. More specifically, we relate the set of associated primes of the tth power $I(\mathcal{C})^t$ to that of the tth power of minors of $I(\mathcal{C})$ and that of mth powers $I(\mathcal{C})^m$ for $m < t$. To do this, we pass to a colon ideal $I(\mathcal{C})^t : M$, where M is a product of distinct variables in R, and then relate this with smaller powers of $I(\mathcal{C})$ and powers of minors of $I(\mathcal{C})$. Our method in proving Theorem 4.6 is to use polarization. In particular, we develop a correspondence (which is not necessary one-to-one) between the associated primes of $I(\mathcal{C})^t$ to that of its polarization.

2. Background and Definitions

A clutter can also be viewed as a simple hypergraph. For this reason, we shall introduce terminology from both clutter and hypergraph languages, and sometimes use these terminology interchangeably.

Throughout the paper, $\mathcal{C}$ will denote a clutter over d vertices $\{x_1, \dots, x_d\}$ and $R = k[x_1, \dots, x_d]$ will be the corresponding polynomial ring. A vertex $x \in V(\mathcal{C})$ is called an *isolated* vertex if $\{x\} \in E(\mathcal{C})$. By definition, if x is an isolated vertex of $\mathcal{C}$ then $\{x\}$ is the only edge in $\mathcal{C}$ that contains x. A clutter is *uniform* if all of its edges have the same cardinality.

A *transversal* (or *vertex cover*) of $\mathcal{C}$ is a set of vertices that has nonempty intersection with all of the edges. We will primarily be interested in minimal transversals (or minimal vertex covers), where minimality is with respect to inclusion. It is easy to see that there is a one-to-one correspondence between minimal transversals of $\mathcal{C}$ and minimal primes of $I(\mathcal{C})$. Since $I(\mathcal{C})$ is a monomial ideal, all minimal primes are monomial ideals, that is, they are generated by subsets of the variables. The minimum cardinality of a transversal of $\mathcal{C}$ will be denoted by $\alpha_0(\mathcal{C})$. Note that by the above correspondence, $\alpha_0(\mathcal{C})$ is also the height of $I(\mathcal{C})$.

A *matching* (or *independent set*) of $\mathcal{C}$ is a set of pairwise disjoint edges. We will refer to generators of a square-free monomial ideal I as being independent if the corresponding edges of the associated clutter are independent; that is, the generators have disjoint *support*. We will primarily be interested in maximal matchings, where maximality is with respect to inclusion. The maximum cardinality of a matching in $\mathcal{C}$ will be denoted by $\beta_1(\mathcal{C})$. Clearly, $\alpha_0(\mathcal{C}) \geq \beta_1(\mathcal{C})$ for any clutter $\mathcal{C}$. A clutter $\mathcal{C}$ is said to satisfy the *König property* if $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$. An ideal is said to satisfy the König property if its associated clutter satisfies the property.

There are two operations commonly used on a clutter $\mathcal{C}$ to produce a new, related, clutter on a smaller vertex set. For a vertex $x \in V(\mathcal{C})$, the *deletion* $\mathcal{C} \setminus x$ is formed by removing x from the vertex set and deleting any edge in $\mathcal{C}$ that contains x. This has the effect of setting $x = 0$, or of passing to the ideal $(I(\mathcal{C}), x)/(x)$ in the quotient ring $R/(x)$. The *contraction* $\mathcal{C}/x$ is obtained by $V(\mathcal{C}/x) = V(\mathcal{C}) \setminus \{x\}$ and $E \in E(\mathcal{C}/x)$ if $x \notin E$ and either $E \in E(\mathcal{C})$ or $E \cup \{x\} \in E(\mathcal{C})$. This process has the effect of setting $x = 1$, or of passing to the localization $I(\mathcal{C})_x$ in R_x. Any clutter formed by a sequence of deletions and contractions is called a *minor* of $\mathcal{C}$. The edge ideal of a minor of $\mathcal{C}$ is also called a minor of $I(\mathcal{C})$. As observed, minors of an edge ideal can be obtained by taking a sequence of quotients and localizations of the original ideal.

A clutter $\mathcal{C}$ is said to have the *packing* property if $\mathcal{C}$ and all of its minors satisfy the König property. We say an ideal has the packing property if its associated clutter has the packing property.

On a more algebraic note, we will need to use the minimal primes, associated primes and symbolic powers of ideals. A prime P is *minimal* over an ideal I if $I \subseteq P$ and there does not exist a prime $Q \neq P$ with $I \subseteq Q \subsetneq P$. A prime P is an *associated* *prime* of I if there exists an element c in R such that $P = (I : c)$. Note that all minimal primes are also associated primes. An ideal I has a primary decomposition

$$I = q_1 \cap \ldots \cap q_t \cap Q_1 \cap \ldots \cap Q_s$$

were q_i and Q_j are primary ideals with $\sqrt{q_i}$ the minimal primes of I. The primes $\sqrt{Q_j}$ are the *embedded* associated primes of I. Observe that if $I = I(\mathcal{C})$ then minimal primes of I correspond to minimal vertex covers of $\mathcal{C}$.

The tth *symbolic power* of an ideal I, denoted by $I^{(t)}$, is the intersection of the primary components of I^t that correspond to minimal primes of I. An ideal I is called *normally torsion-free* if $I^t = I^{(t)}$ for all $t \geq 1$. An ideal I is *unmixed* if all of its minimal primes have the same height. In the language of combinatorial optimization, if $I = I(\mathcal{C})$, this is equivalent to requiring that the *Alexander dual* (or *transversal*) clutter of $\mathcal{C}$ is uniform. A clutter $\mathcal{C}$ is *unmixed* if $I(\mathcal{C})$ is unmixed, i.e., if all minimal vertex covers of $\mathcal{C}$ has the same cardinality.

Suppose I is a square-free monomial ideal minimally generated by $(\mathbf{x}^{\mathbf{a}_1}, \mathbf{x}^{\mathbf{a}_2}, \ldots, \mathbf{x}^{\mathbf{a}_t})$ where $\mathbf{x}^{\mathbf{v}}$ is an abbreviation for $x_1^{v_1} x_2^{v_2} \cdots x_d^{v_d}$. The *incidence matrix* A of I is the matrix whose *ith* column is $\mathbf{a}_i$. The ideal I, or the clutter $\mathcal{C}$ associated to I, satisfies the *max-flow min-cut* (MFMC) property if for all nonnegative integral vectors $\mathbf{w} \in \mathbb{Z}^d$, both sides of the dual linear programming system

$$\min\{\langle \mathbf{w}, \mathbf{v} \rangle \mid \mathbf{v} \geq 0,\, \mathbf{v}A^T \geq \mathbf{1}\} = \max\{\langle \mathbf{y}, \mathbf{1} \rangle \mid \mathbf{y} \geq 0,\, A\mathbf{y} \leq \mathbf{w}\}$$

have integral optimal solution vectors $\mathbf{v}$ and $\mathbf{y}$. Here $\mathbf{1}$ refers to the vector all of whose entries are 1, and $\langle \cdot, \cdot \rangle$ is the standard inner product.

The packing property can also be restated in terms of the dual linear programming system. An ideal has the packing property if and only if the dual linear programming system as above has integral optimal solutions for all $(0, 1, \infty)$-vectors $\mathbf{w}$, that is, when entries of $\mathbf{w}$ are all $0, 1$ or ∞. Thus, it is clear that the MFMC property implies the packing property.

An important fact that we shall make use of is that localization preserves associated primes. That is, if P is a prime ideal containing I then $P \in \operatorname{Ass}(R/I^t)$ if and only if $PR_P \in \operatorname{Ass}(R_P/(IR_P)^t)$. Note that localizing at P is equivalent to passing to a minor of I, thus the packing property is preserved under localization, as is the unmixed property. This allows us to reduce our problem to investigating when the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_d)$ is an associated or embedded prime of $I(\mathcal{C})^t$.

3. Associated Primes and Unmixed Clutters

In this section, we study the set of associated primes of powers a square-free monomial ideal $I(\mathcal{C})$ whose every minor is normally torsion-free. Our primary focus is to determine when $I(\mathcal{C})$ is not normally torsion-free. That is, when a power $I(\mathcal{C})^t$ has embedded primes. We show that $I(\mathcal{C})^t$ does not have any embedded primes for $t \leq \beta_1(\mathcal{C})$. We also show that if, in addition, $\mathcal{C}$ is unmixed then $I(\mathcal{C})$ is normally torsion-free.

Recall that $R = k[x_1, \dots, x_d]$. We shall start by a folklore result (whose proof is elementary), which we state as a lemma for the ease of referring purpose.

Lemma 3.1. *Let K be an ideal and let x be an element in R. Then the following sequence is exact:*

$$0 \to R/(K : x) \xrightarrow{x} R/K \to R/(K, x) \to 0.$$

The next few lemmas exhibit the behavior of associated primes of monomial ideals when passing to subrings or larger rings obtained deleting or adding variables. Note that associated primes of monomial ideals are again monomial, and are generated by subsets of the variables.

Lemma 3.2. *Let K be a monomial ideal in R. Let x an indeterminate of R such that x does not divide any minimal generator of K. Then there is a one-to-one correspondence between the sets* $\operatorname{Ass}(R/K)$ *and* $\operatorname{Ass}(R/(K, x))$ *given by* $P \in \operatorname{Ass}(R/K)$ *if and only if* $Q = (P, x) \in \operatorname{Ass}(R/(K, x))$.

Proof. Suppose $P \in \operatorname{Ass}(R/K)$. Then there is a monomial $c \in R$ such that $P = (K : c)$. Since x does not divide any minimal generator of K, we may assume that x does not divide c. Clearly, $(P, x) \subseteq ((K, x) : c)$. To see the other inclusion, consider a monomial $f \in ((K, x) : c)$. If $x|f$, then $f \in (P, x)$. If x does not divide f, then since x does not divide c, we have that x does not divide fc. Since fc is a monomial and $(K, x) : c$ is a monomial ideal, we have $fc \in K$, and so $f \in (K : c) = P$.

Now suppose that $Q \in \operatorname{Ass}(R/(K, x))$. Since $x \in (K, x) \subseteq Q$ and Q is generated by a subset of the variables, we can write $Q = (P, x)$ for some prime ideal P. Let $c \in R$ be a monomial such that with $Q = (P, x) = ((K, x) : c)$. If $x|c$, then $((K, x) : c) = R \neq P$, a contradiction. Thus, x does not divide c. Let $y \in P$ be a minimal generator. Then x does not divide y and $yc \in (K, x)$. This, and because (K, x) is a monomial ideal, implies that $yc \in K$. Therefore, $P \subseteq (K : c)$. Conversely, let $g \in (K : c)$ be a minimal monomial generator of $(K : c)$. Since x does not divide any minimal generator of K, we have that x does not divide g. It then follows, since $g \in (K : c) \subseteq ((K, x) : c) = Q$ and x does not divide g, that $g \in P$. □

Lemma 3.3. *Let K be a monomial ideal and let M be a monomial in R. Suppose* $P \in \operatorname{Ass}(R/(K : M))$. *Then* $P \in \operatorname{Ass}(R/K)$.

Proof. Since $P \in \operatorname{Ass}(R/(K : M))$, there exists a monomial $c \in R$ such that $P = ((K : M) : c)$. Since $((K : M) : c) = (K : Mc)$, we have that $P = (K : Mc)$. Thus, $P \in \operatorname{Ass}(R/K)$. □

The next lemma will allow us to concentrate on square-free monomial ideals associated to connected clutters. This will be useful when passing to minors, as the minors of a clutter need not be connected. The result is essentially an extension of the preceding lemma and has been proven elsewhere for special cases (see [11, Corollary 5.6] for the normally torsion-free case and see [1, Lemma 2.1] for the case of the edge ideal of a graph).

Lemma 3.4. *Suppose I is a square-free monomial ideal in $S = k[x_1, \dots, x_t, y_1, \dots, y_s]$ such that $I = I_1S + I_2S$, where $I_1 \subseteq S_1 = k[x_1, \dots, x_t]$ and $I_2 \subseteq S_2 = k[y_1, \dots, y_s]$. Then $P \in \operatorname{Ass}(S/I^n)$ if and only if $P = P_1S + P_2S$, where $P_1 \in \operatorname{Ass}(S_1/I_1^{n_1})$ and $P_2 \in \operatorname{Ass}(S_2/I_2^{n_2})$ with $n_1 + n_2 = n + 1$.*

Proof. Suppose first that $P_i \in \operatorname{Ass}(S_i/I_i^{n_i})$ for $i = 1, 2$, and $P = P_1S + P_2S$. Then there exist monomials $c_i \in S_i$ such that $P_i = (I_i^{n_i} : c_i)$, for $i = 1, 2$. Since $I_i^{n_i}$ is a monomial ideal, P_i is a prime ideal generated by a subset of the variables in S_i. Thus, it can be seen that $c_i \in I_i^{n_i-1} \setminus I_i^{n_i}$ for $i = 1, 2$. Now, if $u \in P_1$ then $uc_1c_2 \in I_1^{n_1} I_2^{n_2-1} S \subseteq I^n$. Similarly, if $v \in P_2$ then $vc_1c_2 \in I^n$. Thus, $P \subseteq (I^n : c_1c_2)$. On the other hand, let $w \in S$ be a monomial such that $wc_1c_2 \in I^n$. Since the variable sets for S_1 and S_2 are disjoint, we have $c_1c_2 \in I^{n-1} \setminus I^n$. Write $w = w_1w_2$ where $w_1 \in S_1$ and $w_2 \in S_2$. Observe that if $w_ic_i \notin I_i^{n_i}$ for $i = 1, 2$ then $wc_1c_2 \notin I^n$, a contradiction. Therefore, $w_i \in P_i$ for some i and so $w \in P$.

For the converse, suppose $P \in \operatorname{Ass}(S/I^n)$. Observe again that P is generated by a subset of the variables in S, and so we can write $P = P_1S + P_2S$, where $P_1 = P \cap S_1$ and $P_2 = P \cap S_2$. Also, there exists a monomial $c \in S$ such that $P = (I^n : c)$. As above, it can be seen that $c \in I^{n-1} \setminus I^n$. Write $c = c_1c_2$, where $c_1 \in S_1$ and $c_2 \in S_2$ are monomials. Then $c_1 \in I_1^k$ and $c_2 \in I_2^s$ for some $0 \leq k, s \leq n - 1$ with $k + s = n - 1$. Suppose x is a minimal generator of P_1. Then $x \in P = (I^n : c)$, so $xc_1c_2 \in I^n = I^{k+s+1}$. This implies that $xc_1 \in I_1^{k+1}$. Therefore, $P_1 \subseteq (I_1^{k+1} : c_1)$. On the other hand, let u be a monomial in $(I_1^{k+1} : c_1)$. Then $uc_1c_2 \in I_1^{k+1} I_2^s S = I^n$. This implies that $u \in P$. It follows that $u \in P \cap S_1 = P_1$. Therefore, $P_1 = (I_1^{k+1} : c_1)$. A similar argument shows that $P_2 = (I_2^{s+1} : c_2)$. The conclusion follows by setting $n_1 = k + 1$ and $n_2 = s + 1$. □

As observed before, associated primes behave well under localization, and so our problem can be reduced to examining when the maximal ideal $\mathfrak{m} = (x_1, \dots, x_d)$ is an associated prime of $I(\mathcal{C})^t$.

Proposition 3.5. *Let I be a square-free monomial ideal such that every minor of I is normally torsion-free. Let $y_1, \dots, y_s$ be distinct variables in R, and let $\mathfrak{m} = (x_1, \dots, x_d)$ be the maximal homogeneous ideal of R. Then $\mathfrak{m} \in \operatorname{Ass}(R/I^t)$ if and only if $\mathfrak{m} \in \operatorname{Ass}(R/(I^t : \prod_{i=1}^s y_i))$.*

Proof. By repeated use of Lemmas 3.2 and 3.4, we may assume that the associated clutter of I does not contain any isolated vertices. That is, we may assume that all minimal generators of I are of degree at least 2. Note also that by the established direction of Conjecture 1.1, our hypothesis implies that every minor of I satisfies the packing property.

It follows from Lemma 3.3 that if $\mathfrak{m} \in \operatorname{Ass}(R/(I^t : \prod_{i=1}^s y_i))$ then $\mathfrak{m} \in \operatorname{Ass}(R/I^t)$. We shall use induction on s to prove the other direction. By Lemma 3.1, we have the following exact sequence:

$$0 \to R/(I^t : y_1) \to R/I^t \to R/(I^t, y_1) \to 0.$$

It then follows from [8, Theorem 6.3] that

$$\operatorname{Ass}(R/I^t) \subseteq \operatorname{Ass}(R/(I^t : y_1)) \cup \operatorname{Ass}(R/(I^t, y_1)). \tag{3.1}$$

Let J be the minor of I formed by deleting y_1. That is, the generators of J are obtained from the generators of I by setting $y_1 = 0$. By abusing of notation, we write J^t for both the ideal J^t in $R/(y_1)$ and its extension in R. Note that $J^t \subseteq I^t$, and the generators of J^t are precisely the generators of I^t that are not divisible by y_1. Thus, $(I^t, y_1) = (J^t, y_1)$.

By the hypothesis, J is normally torsion-free, and so $\operatorname{Ass}(R/J^t) = \operatorname{Min}(R/J)$. It follows, since J is square-free, that the maximal homogeneous ideal of $R/(y_1)$ is not an associated prime of J^t unless J consists of isolated vertices. Yet, isolated vertices of J are also isolated vertices of I, and so we may assume that J does not have isolated vertices. Also, by Lemma 3.2, we have $P \in \operatorname{Ass}(R/(J^t, y_1))$ if and only if $P = (P_1, y_1)$ where $P_1 \in \operatorname{Ass}(R/J^t) = \operatorname{Min}(R/J)$. Therefore, if $P = (P_1, y_1) \in \operatorname{Ass}(R/(I^t, y_1)$ then P_1 is not the maximal ideal in $R/(y_1)$. That is, $\mathfrak{m} \notin \operatorname{Ass}(R/(I^t, y_1))$. It now follows from (3.1) that if $\mathfrak{m} \in \operatorname{Ass}(R/I^t)$ then $\mathfrak{m} \in \operatorname{Ass}(R/(I^t : y_1))$.

Suppose now that the assertion is proved for $s-1$, and $\mathfrak{m} \in \operatorname{Ass}(R/I^t)$. Let $M = \prod_{i=1}^{s-1} y_i$. By induction, $\mathfrak{m} \in \operatorname{Ass}(R/(I^t : M))$. By Lemma 3.1, we have the exact sequence

$$0 \to R/((I^t : M) : y_s) \to R/(I^t : M) \to R/((I^t : M), y_s) \to 0.$$

By using [8, Theorem 6.3] again, we have

$$\operatorname{Ass}(R/(I^t : M)) \subseteq \operatorname{Ass}(R/((I^t : M) : y_s)) \cup \operatorname{Ass}(R/((I^t : M), y_s)). \tag{3.2}$$

Let K be the minor of I formed by setting $y_s = 0$. We shall first show that

$$((I^t : M), y_s) = ((K^t : M), y_s).$$

Indeed, consider a monomial $f \in ((I^t : M), y_s)$. If $y_s | f$, then $f \in ((K^t : M), y_s)$. If y_s does not divide f, then $f \in (I^t : M)$, and so $fM \in I^t$. Observe that y_s does not divide M nor f, so y_s does not divide fM. Also, the generators of K^t are generators of I^t that are not divisible by y_s. Thus, $fM \in K^t$. That is, $f \in (K^t : M) \subseteq ((K^t : M), y_s)$. Conversely, consider a monomial $g \in ((K^t : M), y_s)$. If $y_s | g$, then $g \in ((I^t : M), y_s)$. If y_s does not divide g, then $g \in (K^t : M)$, i.e. $gM \in K^t \subseteq I^t$. Thus, $g \in (I^t : M) \subseteq ((I^t : M), y_s)$.

By Lemma 3.2, $P \in \operatorname{Ass}(R/((K^t : M), y_s))$ if and only if $P = (P_1, y_s)$ for some $P_1 \in \operatorname{Ass}(R/(K^t : M))$. By Lemma 3.3, $\operatorname{Ass}(R/(K^t : M)) \subseteq \operatorname{Ass}(R/K^t)$. Also, since K is a minor of I, our hypothesis implies that K is normally torsion-free. That is, $\operatorname{Ass}(R/K^t) = \operatorname{Min}(R/K)$. Thus, by a similar argument as above, we have that $\mathfrak{m} \notin \operatorname{Ass}(R/((K^t : M), y_s)) = \operatorname{Ass}(R/((I^t : M), y_s))$. This and (3.2) imply that $\mathfrak{m} \in \operatorname{Ass}(R/((I^t : M) : y_s)) = \operatorname{Ass}(R/(I^t : \prod_{i=1}^{s} y_i))$. The result is proved. □

As a consequence of Proposition 3.5, we obtain a lower bound for the least power t such that $I(\mathcal{C})^t$ has embedded primes. Notice that associated primes localize, so if

P is an embedded prime of $I(\mathcal{C})^t$ that does not contain any other embedded primes, then we can localize at P and reduce to the case that P is the maximal ideal.

Corollary 3.6. *Let $\mathcal{C}$ be a clutter. Assume that every minor of $I(\mathcal{C})$ is normally torsion-free. If $\mathfrak{m} \in \operatorname{Ass}(R/I(\mathcal{C})^t)$ then $t \geq \beta_1(\mathcal{C}) + 1$.*

Proof. For simplicity of notation, let $\beta = \beta_1(\mathcal{C})$ and $I = I(\mathcal{C})$. By Proposition 3.5, $\mathfrak{m}$ is associated to I^t only if $\mathfrak{m}$ is associated to $(I^t : \prod_{i=1}^d x_i)$, where the product is taken over all distinct variables in R.

Let $\{E_1, \ldots, E_\beta\}$ be a matching in $\mathcal{C}$. Then $\prod_{i=1}^d x_i$ is divisible by $\prod_{j=1}^\beta x^{E_j} \in I^\beta$, so $\prod_{i=1}^d x_i \in I^\beta$. Thus, for $t \leq \beta_1(\mathcal{C})$, we have $(I^t : \prod_{i=1}^d x_i) = R$. Hence, for $t \leq \beta_1(\mathcal{C})$, $\mathfrak{m}$ is not an associated prime of $(I^t : \prod_{i=1}^d x_i)$, and so $\mathfrak{m}$ is not an associated prime of I^t. ◻

Remark 3.7. We will see later, in Theorem 4.6, that the bound in Corollary 3.6 is sharp.

In the rest of this section, we will focus our attention to unmixed clutters. Our next result provides a better control over the colon ideal appearing in Proposition 3.5.

Proposition 3.8. *Let $\mathcal{C}$ be an unmixed clutter satisfying the packing property, and let I be its edge ideal. Let $\{E_1, \ldots, E_{\beta_1(\mathcal{C})}\}$ be a maximal matching in $\mathcal{C}$, and let $g_i = x^{E_i}$ for $i = 1, \ldots, \beta_1(\mathcal{C})$. If $t > \beta_1(\mathcal{C})$ and $I^{t-\beta_1(\mathcal{C})} = I^{(t-\beta_1(\mathcal{C}))}$, then $(I^t : \prod_{i=1}^{\beta_1(\mathcal{C})} g_i) = I^{t-\beta_1(\mathcal{C})}$.*

Proof. For simplicity of notation, let $\beta = \beta_1(\mathcal{C})$ and $M = \prod_{i=1}^\beta g_i$. It is easy to see that $(I^t : M) \supseteq I^{t-\beta}$. To prove the other inclusion, consider a monomial $h \in (I^t : M)$. That is, $hM \in I^t$. Then there exist $F_1, \ldots, F_t \in I$ and $L \in R$ such that $hM = LF_1 \cdots F_t$.

Let P be a minimal prime of I. Since $\mathcal{C}$ is unmixed (equivalently, I is unmixed), we have $\operatorname{ht} P = \alpha_0(\mathcal{C})$. Since $\mathcal{C}$ satisfies the packing property, this implies that $\operatorname{ht} P = \beta$. Also, P covers each of the g_i's. Thus, by the pigeonhole principle, P contains precisely one variable from each g_i for $i = 1, \ldots, \beta$. This implies that $M \in P^\beta \backslash P^{\beta+1}$. Moreover, P also covers F_i for $i = 1, \ldots, t$, and so $hM \in P^t$. Thus, we must have $h \in P^{t-\beta}$.

Now observe that I_P is a complete intersection, and that $I_P = P_P$. Thus, we have $(I^r)_P = (I_P)^r = P_P^r$. This is true for any power r. By our hypothesis, $I^{t-\beta} = I^{(t-\beta)}$. That is, $I^{t-\beta}$ has no embedded primes. It follows that the primary decomposition of $I^{t-\beta}$ has the form

$$I^{t-\beta} = \bigcap_{\sqrt{Q} \in \operatorname{Min}(R/I)} Q.$$

Localizing at a minimal prime P, we get $P_P^{t-\beta} = I_P^{t-\beta} = Q_P$, where Q is the primary ideal associated to P in the above decomposition. This implies that $Q = P^{t-\beta}$. As a consequence, $h \in Q$. This is true for any Q in the primary decomposition of $I^{t-\beta}$. Therefore, $h \in I^{t-\beta}$. Hence, $(I^t : M) \subseteq I^{t-\beta}$ and the result is proved. □

We are now ready to state our result toward Conjecture 1.1 for unmixed clutters.

Theorem 3.9. *Let $\mathcal{C}$ be an unmixed clutter satisfying the packing property. Assume that every minor of $I(\mathcal{C})$ is normally torsion-free. Then $I(\mathcal{C})$ is normally torsion-free.*

Proof. For simplicity of notation, again let $\beta = \beta_1(\mathcal{C})$ and $I = I(\mathcal{C})$. Suppose by contradiction that I is not normally torsion-free. That is, there exists t such that I^t has embedded primes. We choose t minimal with respect to this property. Suppose P is an embedded prime of I^t. Since associated prime localize and all minors of I are normally torsion-free, we may assume that $P = \mathfrak{m}$.

By Corollary 3.6, we have $t > \beta$. Let $\{E_1, \ldots, E_\beta\}$ be a maximal matching in $\mathcal{C}$, and let $g_i = x^{E_i}$. After a reindexing of the variables, we may also assume that $x_1, \ldots, x_s$ are variables in $\prod_{i=1}^{\beta} g_i$. That is, $\prod_{i=1}^{\beta} g_i = \prod_{i=1}^{s} x_i$. By Proposition 3.5, $\mathfrak{m} \in \operatorname{Ass}(R/I^t)$ if and only if $\mathfrak{m} \in \operatorname{Ass}(R/(I^t : \prod_{i=1}^{s} x_i)) = \operatorname{Ass}(R/(I^t : \prod_{i=1}^{\beta} g_i))$. Moreover, by the choice of t, $I^{t-\beta} = I^{(t-\beta)}$. Thus, it follows from Proposition 3.8 that $(I^t : \prod_{i=1}^{\beta} g_i) = I^{t-\beta}$. Now, also by the choice of t, $\mathfrak{m} \notin \operatorname{Ass}(R/I^{t-\beta})$. Therefore, $\mathfrak{m} \notin \operatorname{Ass}(R/I^t)$, which is a contradiction. The result is proved. □

As a direct consequence of Theorem 3.9, we obtain the following result.

Corollary 3.10. *A minimal counterexample to the Conforti-Cornuéjols conjecture cannot be unmixed.*

Remark 3.11. We, in fact, can make Corollary 3.10 stronger. A careful examination of the proof of Proposition 3.8 shows that if there exists a minimal generator g of $I(\mathcal{C})$ such that for each minimal prime P of $I(\mathcal{C})$, only one generator of P divides g, then if $t \geq 2$ and $I(\mathcal{C})^{t-1} = I(\mathcal{C})^{(t-1)}$, then $I(\mathcal{C})^t : g = I(\mathcal{C})^{t-1}$. Thus, following a similar line of arguments as in the proof of Theorem 3.9, if all minors of $I(\mathcal{C})$ are normally torsion-free then $I(\mathcal{C})$ is normally torsion-free. Thus, if a minimal counterexample $\mathcal{C}$ to Conjecture 1.1 existed, then every minimal generator of $I(\mathcal{C})$ must be an element of P^2 for some minimal prime P of $I(\mathcal{C})$.

Example 3.12. Due to our remark above, one might hope that the packing property would imply the existence of a minimal generator g such that $g \in P \backslash P^2$ for all minimal primes P of I. However, while this appears to be true for graphs (* Note to Tai: I think this is true since PP is equivalent to bipartite and I think the double-covers can be used to result in an odd cycle, but I haven't yet worked out all of the details. If this path turns out to be interesting, I can come back and finish this. *) it need not be true for general square-free monomial ideals. For example, let I be the ideal of $k[x_1, \ldots, x_6]$ generated by $I = (x_1x_2x_3, x_4x_5x_6, x_1x_2x_4, x_2x_3x_6, x_1x_4x_5, x_3x_5x_6)$. Then I satisfies the packing property, but $P_1 = (x_1, x_3, x_5)$ and $P_2 = (x_2, x_4, x_6)$ are both minimal primes of I, and for each generator g of I, there is an $i \in \{1, 2\}$ such that $g \in P_i^2$.

4. Polarization and Embedded Associated Primes

In this section, we focus on square-free monomial ideals I whose every minor is normally torsion-free but I is not. We show that in this case, $I^{\beta+1}$ has embedded primes, where β is the matching number of the clutter associated to I. This further shows that the bound given in Corollary 3.6 is sharp.

Throughout the section, $I \subseteq R = k[x_1, \ldots, x_d]$ will denote a square-free monomial ideal whose every minor is normally torsion-free. Our method in this section is to use *polarization*. The process of polarization replaces a power x_i^t by a product of t new variables $x_{(i,1)} \cdots x_{(i,t)}$. We call $x_{(i,j)}$ a *shadow* of x_i. Thus, the polarization of a power I^t of I is a square-free monomial ideal in $d \cdot t$ variables. We will use $\widetilde{I^t}$ to denote the polarization of I^t and use S_t for the new polynomial ring. Observe that if $x_{(i,j)}$ divides a minimal generator M of $\widetilde{I^t}$, then $x_{(i,k)}$ divides M for all $1 \le k \le j$. The *depolarization* of an ideal in S_t is formed by setting $x_{(i,j)} = x_i$ for all i, j. Note that the depolarization of $\widetilde{I^t}$ is I^t.

We begin with a lemma showing that a minimal prime of $\widetilde{I^t}$ cannot contain more than one variables that are shadows of the same variable in R. This will restrict the class of primes to be considered when dealing with polarization.

Lemma 4.1. *Let P is a minimal prime of $\widetilde{I^t}$ in S_t, and suppose $x_{(i,j)} \in P$. Then $x_{(i,k)} \notin P$ for all $k \neq j$.*

Proof. Let $\mathcal{C}_t$ be the associated clutter of $\widetilde{I^t}$. Then P is a minimal vertex cover of $\mathcal{C}_t$. Suppose by contradiction that $x_{(i,j)}$ and $x_{(i,k)}$ are both in P and $k \neq j$. Without loss of generality, assume $k < j$. Let v be a minimal generator of $\widetilde{I^t}$ that is covered by $x_{(i,j)}$. From our observation above, v is divisible by $x_{(i,l)}$ for all $l \le j$. In particular, v is divisible by $x_{(i,k)}$. Thus, $P \setminus \{x_{(i,j)}\}$ is a vertex cover of $\mathcal{C}_t$. This is a contradiction to the minimality of P. The lemma is proved. ◻

Remark 4.2. Observe that every minimal prime of I lifts to a minimal prime of the polarization $\widetilde{I^t}$ of I^t for every t. Indeed, if $(x_1, \ldots, x_r)$ is a minimal prime of I, then $(x_1, \ldots, x_r)$ is a minimal prime of I^t. This implies that $\{x_{(1,1)}, \ldots x_{(r,1)}\}$ is a vertex cover for the clutter associated to $\widetilde{I^t}$. This cover is necessarily minimal. In other words, $(x_{(1,1)}, \ldots, x_{(r,1)})$ is a minimal prime of $\widetilde{I^t}$.

Our next lemma shows that the embedded primes of I^t also lift to associated primes of $\widetilde{I^t}$, and that associated primes of the $\widetilde{I^t}$ depolarize to associated primes of I^t. This creates a correspondence, which is not usually one-to-one, between associated primes of I^t and associated primes of its polarization $\widetilde{I^t}$.

Lemma 4.3. *Let I be as above, and let t be a positive integer.*

(1) *Let $P \in \operatorname{Min}(R/\widetilde{I^t})$, and let p be the depolarization of P. Then $p \in \operatorname{Ass}(R/I^t)$.*

(2) *Let* $q \in \operatorname{Ass}(R/I^t)$. *Then, there is at least one prime* $Q \in \operatorname{Ass}(R/\widetilde{I^t})$ *such that the depolarization of* Q *is* q.

Proof. (1) By definition, there exists a monomial c in S_t such that $P = (\widetilde{I^t} : c)$. Let $\mathcal{C}_t$ be the clutter associated to $\widetilde{I^t}$. Since $\widetilde{I^t}$ is square-free, we may assume that c is square-free and $c \notin P$. Thus, we may take c to be the product of variables of S_t that are not in P. Since there are subsets of $\{x_1, \ldots, x_d\}$ that are minimal primes of I, it follows from Remark 4.2 that the set $\{x_{(1,1)}, x_{(2,1)}, \ldots, x_{(d,1)}\}$ cannot be contained in any minimal prime of $\widetilde{I^t}$. In particular, there exists an i such that $x_{(i,1)}$ is not in P. That is, $x_{(i,1)}$ divides c.

For a monomial M in S_t, define $\overline{M}$ to be the maximal monomial divisor of M with the property that if $x_{(l,j)}$ divides $\overline{M}$ for some l and j, then $x_{(l,k)}$ divides $\overline{M}$ for all $k \leq j$. It can be seen from the definition that $\overline{M}$ is the polarization of some monomial in R. Observe that since $x_{(i,1)}$ divides c, $\overline{c}$ is non-trivial.

Let c_1 be the depolarization of $\overline{c}$. Then $\widetilde{c_1} = \overline{c}$ divides c. Thus $c_1 \notin I^t$. Suppose $P = \{x_{(i_1,j_1)}, \ldots, x_{(i_r,j_r)}\}$. Consider $x_{(i_1,j_1)} \in P$. We have $x_{(i_1,j_1)}c \in \widetilde{I^t}$, and so there is a monomial generator v_1 of I^t and an monomial $f_1 \in S_t$ such that $x_{(i_1,j_1)}c = f\widetilde{v_1}$, where $\widetilde{v_1}$ is the polarization of v_1 in S_t. Moreover, by the definition of polarization and by the maximality of $\overline{c}$, $\widetilde{v_1}$ must divide $\overline{x_{(i_1,j_1)}c}$. Thus the depolarization of $\overline{x_{(i_1,j_1)}c}$ is in I^t. This element is of the form $x_{i_1}^s c_1$ for some s. Choose s_{i_1} minimal such that $x_{i_1}^{s_{i_1}} c_1 \in I^t$. Since $c_1 \notin I^t$, $s_{i_1} > 0$. Set $c_2 = x_{i_1}^{s_{i_1}-1} c_1$. Then, $c_2 \notin I^t$, $x_{i_1}c_2 \in I^t$, and $\widetilde{c_2}$ divides c.

Consider $x_{(i_2,j_2)} \in P$. As above, we have $x_{(i_2,j_2)}c \in \widetilde{I^t}$, and so $x_{(i_2,j_2)}c = f_2\widetilde{v_2}$ for some minimal generator $v_2 \in I^t$ and some $f_2 \in S_t$. By a similar line of arguments as above, $\widetilde{v_2}$ divides $\overline{x_{(i_2,j_2)}c}$, and the depolarization of $x_{i_2}^s c_2$ of $\overline{x(i_2,j_2)c}$ is in I^t. Take s_{i_2} minimal so that $x_{i_2}^{s_{i_2}} c_2 \in I^t$. Again, note that $s_{i_2} > 0$. Set $c_3 = x_{i_2}^{s_{i_2}-1} c_2$. Then $c_3 \notin I^t$, $x_{i_2}c_3 \in I^t$, and $\widetilde{c_3}$ divides c. Notice further that $x_{i_1}c_3 \in I^t$, since $x_{i_1}c_2 \in I^t$ and c_2 divides c_3.

By repeating this process, we may assume that c_r has been defined so that $c_r \notin I^t$, $x_{i_l}c_r \in I^t$ for $1 \leq l \leq r$, and $\widetilde{c_r}$ divides c. Thus $p \subseteq (I^t : c_r)$. To see that the equality holds, consider a monomial $M \in (I^t : c_r) \setminus p$. Without loss of generality, we may assume M is minimal in the sense that no proper divisor of M satisfies this condition. Since the highest power of each variable appearing in minimal generators of I^t is t, the minimality of M implies that the power of each variable appearing in Mc_r is at most t. Thus, polarization of Mc_r makes sense in S_t, and we have $\widetilde{Mc_r} \in \widetilde{I^t}$. Observe that $\widetilde{Mc_r}$ can be written as $\widehat{M}\widetilde{c_r}$ for some $\widehat{M} \in S_t$ that depolarizes to M. Since $\widetilde{c_r}$ divides c, we have that $\widehat{M}c \in \widetilde{I^t}$. It follows that $\widehat{M} \in P$. That is, $x_{(i_k,j_k)}$ divides $\widehat{M}$ for some $1 \leq k \leq r$. Therefore, x_{i_k} divides M for some $1 \leq k \leq r$, and so $M \in p$.

(2) By definition, there exists a monomial $b \in R \setminus q$ such that $q = (I^t : b)$. Suppose $x_i \in q$ and $s_i \geq 0$ is maximal such that $x_i^{s_i}$ divides b. Let v be a minimal generator

of I^t and let f be a monomial in R such that $x_ib = fv$. If x_i divided f, then $b \in I^t$, which is a contradiction. Thus, $x_i^{s_i+1}$ divides v, and in particular, $s_i + 1 \leq t$. Therefore, the polarization of x_ib exists in S_t, and $\widetilde{x_ib} = \widetilde{fv} \in \widetilde{I^t}$. By the choice of s_i, $x_{(i,s_i+1)}\widetilde{b} \in \widetilde{I^t}$. Therefore, $x_{(i,s_i+1)} \in (\widetilde{I^t} : \widetilde{b})$. Let Q be the prime ideal of S_t generated by $\{x_{(i,s_i+1)} \mid x_i \in q\}$.

We have seen that $Q \subseteq (\widetilde{I^t} : \widetilde{b})$ and Q depolarizes to q. We shall show that $Q = (\widetilde{I^t} : \widetilde{b})$. Indeed, consider a monomial $z \in (\widetilde{I^t} : \widetilde{b})$. Then, $z\widetilde{b} \in \widetilde{I^t}$. That is, $z\widetilde{b}$ is a multiple of a generator of $\widetilde{I^t}$. Since the generators of $\widetilde{I^t}$ are the polarizations of the generators of I^t, there exists a minimal generator w of I^t and $h \in S_t$ such that $z\widetilde{b} = h\widetilde{w}$. We may assume that $(z, h) = 1$. Let y be the depolarization of z. Then yb is a multiple of w. Thus, $y \in (I^t : b) = q$. Thus x_i divides y for some $x_i \in q$, and so $x_{(i,j)}$ divides z for some j.

Observe that if $j \leq s_i$, then $x_{(i,j)}^2$ divides $z\widetilde{b}$ and so since the minimal generators of $\widetilde{I^t}$ are square-free, $x_{(i,j)}$ divides h, a contradiction to our assumption that $(z, h) = 1$. Suppose that $j \geq s_i + 2$. Then by the definition of s_i, $x_{(i,s_i+1)}$ does not divide $z\widetilde{c}$. Therefore, $x_{(i,s_i+1)}$ does not divide $h\widetilde{w}$. This together with the definition of polarization and the fact that $\widetilde{w}$ is a minimal generator of $\widetilde{I^t}$ imply that $x_{(i,l)}$ does not divide $\widetilde{w}$ for $l \geq s_i + 1$. In particular, $x_{(i,j)}$ does not devide $\widetilde{w}$. Thus, $x_{(i,j)}$ divides h, again a contradiction to our assumption that $(z, h) = 1$. Thus, $j = s_i + 1$, and we have $x_{(i,s_i+1)}$ divides z, and so $z \in Q$. This is true for any monomial $z \in (\widetilde{I^t} : \widetilde{b})$, so $Q \supseteq (\widetilde{I^t} : \widetilde{b})$. The result is proved. □

Remark 4.4. Notice that in the proof of Lemma 4.3, Q was determined by q and by a fixed b. Thus, Q need not be a unique minimal prime of $R/\widetilde{I^t}$ corresponding to q.

Example 4.5. Let $R = k[x, y, z]$ and let $I = (xy, yz, xz)$ be the edge ideal of a triangle. Then $I^2 = (x^2y^2, y^2z^2, x^2z^2, xy^2z, xyz^2, x^2yz)$ and

$$\widetilde{I^2} = (x_1x_2y_1y_2, y_1y_2z_1z_2, x_1x_2z_1z_2, x_1y_1y_2z_1, x_1y_1z_1z_2, x_1x_2y_1z_1).$$

The associated primes of R/I^2 are $\{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}$ and the associated primes of $S/\widetilde{I^2}$ are

$$\begin{array}{lllll} \{x_1, y_1\}, & \{x_2, y_1\}, & \{x_1, y_2\}, & \{x_1, z_1\}, & \{x_2, z_1\}, \\ \{y_1, z_1\}, & \{y_2, z_1\}, & \{x_1, z_2\}, & \{y_1, z_2\}, & \{x_2, y_2, z_2\}. \end{array}$$

Using the correspondence between associated primes of I^t and of its polarization, we are now ready to prove our main result in this section.

Theorem 4.6. *Let $I \subseteq R = k[x_1, \ldots, x_d]$ be a square-free monomial ideal such that every minor of I is normally torsion-free. Suppose that the associated clutter $\mathcal{C}$ of I does not have the packing property. Then $\mathfrak{m} \in \operatorname{Ass}(R/I^{\beta_1(\mathcal{C})+1})$.*

Proof. For simplicity of notation, let $\beta = \beta_1(\mathcal{C})$. By definition, it can be seen that $\prod_{i=1}^d x_i \in I^\beta \setminus I^{\beta+1}$. Thus no minimal generator of $I^{\beta+1}$ is square-free. This implies that $A = \{x_{(1,2)}, x_{(2,2)}, \ldots x_{(d,2)}\}$ is a vertex cover of the associated clutter $\mathcal{C}'$ of $\widetilde{I^{\beta+1}}$.

We claim that A is a minimal vertex cover of $\mathcal{C}'$. Suppose by contradiction that A is not. Then there is a subset B of A that is a minimal vertex cover of $\mathcal{C}'$. Let Q be the prime ideal generated by elements in B. By Lemma 4.3, Q depolarizes to an associated prime q of I^t. Since every minor of I is normally torsion-free, a localization argument shows that the only possible embedded prime of I^t is the maximal homogeneous ideal $\mathfrak{m}$ of R. This implies that Q depolarizes to a minimal prime of I^t. That is, q is a minimal prime of I^t, and so q is a minimal prime of I. By reindexing the variables in R if necessary, we may assume that $q = (x_1, \ldots, x_s)$. Then $C = \{x_1, \ldots, x_s\}$ is a minimal vertex cover of $\mathcal{C}$. By definition, for each $x_i \in C$, there exists a monomial generator g_i of I such that g_i is not covered by $C \setminus \{x_i\}$. It follows from our hypothesis and the established direction of Conjecture 1.1 that every minor of $\mathcal{C}$ has the packing property. Thus, our hypothesis implies that $\mathcal{C}$ does not have the König property. That is, $\alpha_0(\mathcal{C}) > \beta_1(\mathcal{C})$. As observed before, $\alpha_0(\mathcal{C}) = \operatorname{ht} I$, so $s \geq \alpha_0(\mathcal{C}) \geq \beta + 1$. Observe now that for any $j = 1, \ldots, s$, x_j^2 does not divide $M = \prod_{i=1}^{\beta+1} g_i$. This implies that the polarization $\widetilde{M}$ in $\widetilde{I^{\beta+1}}$ is not covered by $(x_{(1,2)}, \ldots, x_{(s,2)})$, a contradiction to the fact that B is a vertex cover of $\mathcal{C}'$.

We have shown that A is a minimal vertex cover of the clutter $\mathcal{C}'$ associated to $\widetilde{I^{\beta+1}}$. Let P be the ideal generated by elements in A. Then P is a minimal prime of $\widetilde{I^{\beta+1}}$. It then follows from Lemma 4.3 that $\mathfrak{m}$, which is the depolarization of P, is an associated prime of $I^{\beta+1}$. The result is proved. □

***** Not sure if we want to include the following! *****

Theorem 4.6 is also provides a means for searching for embedded primes through localization. For such a prime $P = (I^t : c)$, it is often useful to have a concrete description for the monomial c. Our next results give useful information about such an element, as well as an alternate proof to our bound in Corollary 3.6.

Lemma 4.7. *Let $\mathcal{C}$ be a clutter with edge ideal I. Suppose that every minor of I is normally torsion-free, and $\mathfrak{m} = (I^t : c)$ for some $t \geq 1$. Then, x divides c for every variable x in R. In particular, $c \in I^{\beta_1(\mathcal{C})}$, and so $t \geq \beta_1(\mathcal{C}) + 1$.*

Proof. Suppose there exists a variable x in R that does not divide c. Consider the exact sequence

$$0 \to R/((I^t : c) : x) \to R/(I^t : c) \to (R/((I^t : c), x) \to 0.$$

Since $\mathfrak{m} = (I^t : c)$, we have $xc \in I^t$. Thus, the left module of the exact sequence vanishes. Also, since x does not divide c, by a similar argument as in the proof of Proposition 3.5, we have $((I^t : c), x) = ((J^t : c), x)$, where J is the minor of I formed by setting $x = 0$. Our exact sequence now becomes:

$$0 \to R/\mathfrak{m} \to R/((J^t : c), x) \to 0.$$

This implies that $\mathfrak{m} = ((J^t : c), x)$. It then follows that the maximal homogeneous ideal $\overline{\mathfrak{m}}$ of $R/(x)$ is an associated prime of J^t. However, since I has no isolated vertices, neither does J. Thus, $\overline{\mathfrak{m}}$ is an embedded prime of J. This is a contradiction to the hypothesis that J was normally torsion-free.

Now let $E_1, \dots, E_{\beta_1(\mathcal{C})}$ be a matching in $\mathcal{C}$. Then since these edges are pairwise disjoint and every variable in R divides c, it can be seen that $\prod_{i=1}^{\beta_1(\mathcal{C})} x^{E_i}$ divides c. Thus, $c \in I^{\beta_1(\mathcal{C})}$. Moreover, since $\mathfrak{m} \neq R$, we must have $c \notin I^t$. Therefore, $t \geq \beta_1(\mathcal{C}) + 1$. □

Proposition 4.8. *Let $\mathcal{C}$ be a clutter satisfying the packing property, and let I be its edge ideal. Assume that $\mathcal{C}$ does not consist of all isolated vertices, every minor of I is normally torsion-free, and $\mathfrak{m} = (I^t : c)$. Then $\prod_{i=1}^d x_i$ divides c. Moreover, for every minimal prime P of I with $\operatorname{ht} P = \operatorname{ht} I$, there exists $x \in P$ such that x^2 divides c.*

Proof. The first statement follows from Lemma 4.7. We shall prove the second statement. For simplicity of notation, let $\alpha = \alpha_0(\mathcal{C}) = \operatorname{ht} I$ and $\beta = \beta_1(\mathcal{C})$.

It follows from our hypothesis that $\operatorname{ht} P = \alpha = \beta$. Since $\mathcal{C}$ does not consist of only isolated vertices, $\mathfrak{m}$ is not a minimal prime of $I(\mathcal{C})$. Thus, there exists a variable y in R such that $y \notin P$. Since $\mathfrak{m} = (I^t : c)$, we have that $yc = F_1 \cdots F_t h$ for some minimal generators F_i's of I and $h \in R$. By Lemma 4.7, $t \geq \beta + 1$. Moreover, for any i, $F_i \in P$ (since P is a minimal prime of I so P corresponds to a minimal vertex cover of $\mathcal{C}$). Therefore, there exists a variable $x \in P$ that divides at least two of the F_i's. This implies that x^2 divides yc. Now since $y \notin P$, which implies that $y \neq x$, and so x^2 divides c. □

Notice that for every minimal vertex cover of size α, such an x exists. So unless x is an element of every minimal vertex cover of size α, there must be at least two variables whose squares divide c.

3. We show how to use a generalization of the red/blue arguement to "work outward" from a minimally non-ntf minor.

4. We give special cases: connected in codim 1 was one we talked about, also n variables any d of which formed an edge; special cycles (eg; cycle of length n, pure generators of degree three, eg: abc, bcd, cde, ... until you cycle back, I'm drawing a blank on the name here). In most of the special cases, we showed what was associated (ex: m is associated when ....). I have a collection of these types of results. I don't think we had the entire picture at the time, but we might be able to use our results from the proceeding section to clean it up. Example: if before we could show what did occur and where, we might now be able to argue that we had found it all by localizing and showing that m could not have appeared earlier or something.

5. We had very specific examples such as the open n-simplex where we really could prove everything appeared exactly where we wanted.

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