Black Holes in Spherically Symmetric Dust-Filled Closed Universes

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Abstract

We investigate the problem of defining black holes in closed universes. We do this by first finding all spherically symmetric marginally trapped surfaces in the dust-filled Tolman S^3 universes, and test three definitions which have been proposed for black holes in closed universes. The definitions of Wheeler and Tipler are easy to apply, but we find that Hayward's definition is quite complicated.

1 Introduction

Black holes, originally defined solely in asymptotically flat spacetimes (in particular, the Schwarzschild solution to the Einstein field equations), have yet to find an ideal definition in other possible spacetime geometries. An appropriate definition should distinguish the local properties of the spacetime which are characteristic of black holes from those that are inherent in the global structure

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of spacetime. Several possible definitions have been suggested; here we consider the applicability and practicality of these definitions in the general closed, spherically symmetric dust solution to the field equations.

1.1 Black hole definitions

The notion of a trapped surface - a spacelike surface, the divergence of whose surface area is negative along null geodesics - is integral to the definitions that we will consider. Specifically, for a 2-surface of surface area A, we define a function

$$\theta_\pm = \frac{1}{A}\partial_\pm A \tag{1.1}$$

where the $\partial_\pm$ is the directional derivative in the null, radial direction.

Definition 1.1 (Trapped, marginally trapped surface) *A 2-sphere constitutes a trapped surface if, for all points on the sphere, $\theta_\pm < 0$ for both $+$ and $-$. We define a marginally trapped surface (MTS) as a 2-sphere that satisfies $\theta_\pm < 0$ for one of $+$ or $-$, and $\theta_\pm = 0$ for the other [1].*

Thus a trapped surface has the characteristic that the divergence of its area, even along outwardly directed light rays, is negative. In asymptotically flat spacetimes, the existence of trapped surfaces has long been regarded as the hallmark of a black hole. However, it has been shown [2] that in the collapsing phase of a closed Friedmann universe, every spherically symmetric 2-sphere satisfies the trapped surface criterion sufficiently close to the final singularity. Hence the concept of a trapped surface alone will not suffice to differentiate between local and global collapse. This subtlety has been overlooked by several competent researchers, e.g. [3, 4]. This particular example leads us to the following possible definition, due to Wheeler [5]:

Definition 1.2 (Cosmological, non-cosmological trapped surface) *For a closed universe, a trapped surface will be considered a non-cosmological trapped surface if it occurs before the moment of maximum expansion. That is, it occurs in the region of spacetime that lies causally to the past of the maximal hypersurface. All other trapped surfaces will be considered cosmological trapped surfaces.*

As we will discuss in more detail later, the maximal hypersurface will be unique provided that the spacetime satisfy certain physically reasonable energy conditions set out by Hawking and Ellis in [1].

One can see that this definition, somewhat by brute force, separates out the trapped surfaces that occur solely as a result of the global collapse of the universe from those that are of a local collapse of a star. Another possible definition is the distinction made by Hayward [6]:

Definition 1.3 (Inner, outer, degenerate marginally trapped surfaces) *Let $L_\pm$ be the Lie derivative in one of the two null radial direction; for scalar functions, $L_\pm = \partial_\pm$. Also, let S be the trapped surface in question. Marginally trapped surfaces are classified as follows: if $L_\mp \theta_\pm|_S = \partial_\mp \theta_\pm|_S > 0$, S is called an inner trapped surface; if $L_\mp \theta_\pm|_S = 0$, S is degenerate; if $L_\mp \theta_\pm|_S < 0$, S is called an outer trapped surface.*

That is, this definition classifies trapped surfaces according to the second derivative of their area in the null radial direction (i.e., along the paths of light). Inner MTS are intended to be associated with black holes, and outer MTS are intended to be associated with universal collapse. One final definition to consider is offered by Tipler [7]:

Definition 1.4 (Cosmological, non-cosmological trapped surface) *Let S be a spacelike hypersurface containing a trapped surface T region that is bounded by a marginally trapped surface region ∂T. Along ∂T, $\theta_\pm = 0$ for one of $\pm$, and is < 0 for the other; without loss of generality, assume that it is $= 0$ for $+$, and < 0 for $-$ (otherwise, reverse the following). Let n_+^μ be the null vector on such a MTS, and let t^μ be a spacelike vector on the MTS in ∂T which points* toward *the trapped surface region T. If $n_+^\mu t_\mu < 0$, then a trapped surface in T will be called a non-cosmological trapped surface. Otherwise, it will be called cosmological.*

It is easy to show [7] that this definition works to distinguish between cosmological and non-cosmological trapped surfaces in the closed Friedmann and Schwarzschild cases. That is, in a spacelike slice through a trapped surface region in the collapsing phase of the Friedmann model the null vector that corresponds to the negative value of $\theta_\pm$ points *away from* the trapped surface region, whereas the corresponding null vector in the Schwarzschild solution points *towards* the trapped surface region.

In other words, if the null vector in the direction for which $\theta = 0$ points *toward* the trapped surfaces, the trapped surfaces are cosmological trapped surfaces, and if it points away *away* from the trapped surface region, $(n_+^\mu t_\mu < 0)$, then the MTS is the boundary of a non-cosmological trapped surface region.

2 Location of trapped surfaces

We now apply these possible definitions to spherically symmetric, closed dust universes. We first compute a criterion for the location of trapped surfaces in such a spacetime. The metric for a general spacetime of this form is given in [8]; we shall use their notation. The line element for such a spacetime is given as

$$ds^2 = -dt^2 + \frac{(\frac{\partial Y}{\partial r}(r,t))^2}{1 - f^2(r)}\, dr^2 + Y^2(r,t)[d\theta^2 + \sin^2\theta\, d\phi^2] \tag{2.1}$$

The variables θ (not to be confused with $\theta_\pm$) and ϕ are the standard coordinates on a 2-sphere; hence the spherical symmetry. There are several functions that are hidden in this metric; there exist functions m, f, and t_o which are arbitrary functions of r, and will be used, along with the function $h(\eta)$, in the definition of Y and t. h is given by

$$h(\eta) = \eta - \sin\eta \tag{2.2}$$

where η is related to t and r by

$$t = t_o(r) \pm \frac{h(\eta)m(r)}{f^3(r)} \tag{2.3}$$

Finally, the scale factor Y is given in terms of the arbitrary functions $f(r)$ and $m(r)$ as follows:

$$Y = \frac{\dot{h}(\eta)m(r)}{f^2(r)} = \frac{(1-\cos\eta)m(r)}{f^2(r)} \tag{2.4}$$

where we are making the convention that derivatives with respect to r will be denoted by a prime ($'$), and with respect to η by an overdot ($\cdot$). For this spacetime to satisfy the energy conditions [8], it must be true that $\frac{m'}{Y'} \geq 0$.

2.1 Change of variables

In order to compute the locations of marginally trapped surfaces, we will make a change of variables to a more natural coordinate system. Since we are considering a spacetime which is topologically $S^3 \times R$, a better radial coordinate than r (which only covers half of S^3) is the angular variable χ, defined by $r = \sin\chi$. We now adopt several other conventions. First, we will confuse $f(r)$ and $f(\chi)$; that is, write $f(r) = f(\chi)$, even though actually, $f(r) = f(\sin\chi)$. We do the same for the functions Y, m, and t_o. Then for simplicity later on, call

$$Q(\chi) = \frac{m(\chi)}{f^3(\chi)} \tag{2.5}$$

We will also make a temporal change of variables; η serves as a more natural coordinate than t, since t is defined as a function given in terms of η and χ. We will use this relationship to make the change of variables in the metric. t is the function

$$t = t_o(\chi) \pm \frac{h(\eta)m(\chi)}{f^3(\chi)} \tag{2.6}$$

Thus, we can take the differential of t:

$$dt = t' d\chi + \dot{t} d\eta \tag{2.7}$$

Where here the prime denotes the derivative with respect to χ (We will use the prime to denote both the derivative with respect to r and with respect to χ. The context will clearly show which is meant). Now, substituting into the line element gives us

$$ds^2 = -\dot{t}^2 d\eta^2 - 2\dot{t}t' d\eta d\chi + [\frac{Y'^2}{1-f^2} - t'^2] d\chi^2 + Y^2 [d\theta^2 + \sin^2\theta \, d\phi^2] \tag{2.8}$$

With the identification

$$f(\chi) = \sin\chi, \quad m(\chi) = \frac{a_{max}}{2} \sin^3\chi, \quad t_o = 0 \tag{2.9}$$

the metric reduces to the S^3 Friedmann dust metric, with the scale factor $a(\eta) = (a_{max}/2)(1 - \cos\eta)$,

2.2 Computation of trapped surfaces

We now pick (for any point in the spacetime) a null vector $n_\pm$ whose spatial component is purely radially directed (χ, η components only). Since we are not concerned with normalization, we will choose the vector as $n_\pm = (1, a, 0, 0)$. Thus $n_\pm$ is null, so $n^\mu_\pm n_{\pm\mu} = 0$. From (cite equation), the null condition implies

$$n_\pm = (1, -\frac{\dot{t}}{t' \pm Y'/\sqrt{1-f^2}}, 0, 0) \tag{2.10}$$

We compute the directional derivative $\partial_\pm$ along $n_\pm$ (the null direction in which $n_\pm$ points) to be

$$\partial_\pm = n^\mu_\pm \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \eta} - \frac{\dot{t}}{t' \pm Y'/\sqrt{1-f^2}} \frac{\partial}{\partial \chi} \tag{2.11}$$

That is, since there are two possible radially directed null vectors (given by either $+$ or $-$), we define each derivative dependent upon choice of $+$ or $-$.

Now consider a 2-sphere at any location in the spacetime. Fixing the pair (η, χ) fixes a 2-sphere, since the metric restricted to θ, ϕ is that of a 2-sphere. The surface area of the two-sphere is given by

$$A = 4\pi Y^2 \tag{2.12}$$

since Y is the scale factor for the 2-sphere metric. Using (cite equations) to evaluate (cite equations), we obtain:

$$\theta_\pm = \frac{1}{A} \partial_\pm A = \frac{2}{Y} [\dot{Y} - \frac{\dot{t} Y'}{t' \pm Y'/\sqrt{1-f^2}}] \tag{2.13}$$

Thus we have a computable criterion for the location of marginally trapped surfaces in arbitrary closed, spherically symmetric dust spacetimes; simply set one of $\theta_\pm = 0$, and check that the other is negative in the set of points satisfying that criterion.

We can expand out t in the definition of $\theta_\pm$:

$$\theta_\pm = \frac{2}{Y} [\dot{Y} - \frac{\dot{h} Q Y'}{t'_o \pm h Q' \pm Y'/\sqrt{1-f^2}}] \tag{2.14}$$

Note that the $\pm$ appearing in $t'_o \pm h Q'$ is the $\pm$ appearing in the original definition of t.

In [9], a similar criterion is computed in the original variables (t, χ, θ, ϕ). Marginally trapped surfaces satisfy

$$(\mathring{Y})^2 = 1 - f^2 \tag{2.15}$$

where the overcircle denotes the partial derivative with respect to t.

3 Application of the definitions

Here we will apply each of the possible definitions to the closed spherically symmetric dust model, for which we computed the trapped surfaces above.

3.1 Definition 1: Existence prior to maximum expansion

Now we consider the first definition of a black hole that we employed - those trapped surfaces that exist prior to the maximal hypersurface. To aid us in evaluating this criterion, we have the following

Theorem 3.1 *A spacetime with a compact Cauchy surface which begins and ends in crushing singularities, and which satisfies the strong energy condition, can be uniquely foliated by hypersurfaces defined by the relation Tr K = constant, where K is the extrinsic curvature of the spacetime, and Tr is the trace. Moreover, the hypersurface defined by Tr K = 0 is the unique maximal hypersurface of the spacetime [10, 11, 12].*

Here we find a general formula for the trace of the extrinsic curvature (K) of an arbitrary hypersurface S in the general spherically symmetric dust model. Let N^μ denote the components of the normal vector to the hypersurface, with respect to the basis $(\frac{\partial}{\partial t}, \frac{\partial}{\partial \chi}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi})$. Warning: for the rest of this section alone, we will be using the coordinates (t, χ, θ, ϕ) instead of $(\eta, \chi, \theta, \phi)$. The metric (2.1) is expressed in coordinates comoving with the dust [9]. Therefore, $\partial/\partial t$ is a unit tangent vector to timelike geodesics which are everywhere normal to the spacelike hypersurfaces $t =$ constant. Then one has [(1, p.99)] for the extrinsic curvature $K_{\mu\nu}$:

$$N_{\mu;\nu} = K_{\mu\nu} \tag{3.1}$$

Thus

$$tr\ K = N^{\mu};_{\mu} = \Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}$$

since $N^{\mu} = (1, 0, 0, 0)$. Evaluating the Christoffe symbols in the coordinate basis of (2.1) gives

$$tr\ K = \left(\frac{\mathring{Y}'}{Y'} + \frac{2\mathring{Y}}{Y} \right) \tag{3.2}$$

As a check, we evaluate (3.2) in the dust Friedmann case. Regarding η in (2.4) as a function of proper time t gives

$$tr\ K = \frac{3 \sin \eta}{(1 - \cos \eta)} \frac{d\eta}{dt} \tag{3.3}$$

But in the dust Friedmann model, $t = (a_{max}/2)(\eta - \sin \eta)$, so $d\eta/dt = (dt/d\eta)^{-1} = ([a_{max}/2][1 - \cos \eta])$ and thus

$$tr\ K = \frac{6 \sin \eta}{a_{max}(1 - \cos \eta)^2} \tag{3.4}$$

which is the standard result. We now use this to determine if the trapped surface lies to the past of the maximal hypersurface. If Tr K is positive, then the $tr\ K =$

0 maximal hypersurface will be to the further of the $tr\ K > 0$ hypersurface. Thus we have a *sufficient* condition to determine if the $t =$ constant hypersurface lies to the past of the maximal hypersurface: any trapped surface that lies in a $t =$ constant hypersurface such that $tr\ K > 0$ must necessarily occur *before* the moment of maximal expansion. Thus, if

$$\frac{\mathring{Y}'}{Y'} + \frac{2\mathring{Y}}{Y} > 0 \tag{3.5}$$

every where in a $t =$ constant hypersurface, the hypersurface will satisfy tr $K > 0$, and thus a trapped surface in this $t =$ constant hypersurface will be non-cosmological.

From (2.15), we have that points on marginally trapped surfaces satisfy $\mathring{Y} = \pm\sqrt{1 - f^2}$; we must choose the negative, since the rate of change of surface areas of 2-spheres (which is proportional to Y) is negative when the dust sphere are collapsing. Thus, for marginally trapped surfaces, the requirement that they lie before the maximal hypersurface (i.e., Tr $K > 0$) , and hence be non-cosmological, reduces to

$$\frac{f' f^2}{2mf' - fm'} + 2\frac{1 - f^2}{m} < 0 \tag{3.6}$$

If the marginally trapped surface region lies before the maximal hypersurface, and is thus non-cosmological in origin, clearly the trapped surface region which it bounds must lie before the maximal hypersurface.

Thus a sufficient condition for the maximal hypersurface to lie to the future of the $t =$ constant hypresurface is not difficult to obtain, making it practical to determine (according to Wheeler's definition of a cosmological trapped surface) whether a trapped surface is cosmological or not.

3.2 Definition 2: Null vector directions

On the practical scale, the third definition (Tipler's) is an easy distinction to make. We simply set the spacelike hypersurface to be defined by η = constant, such that that hypersurface contains a trapped surface, and look at the χ component of $n^{\mu}_{\pm}$:

$$n^{\chi}_{\pm} = -\frac{\dot{t}}{t' \pm Y'/\sqrt{1-f^2}} \tag{3.7}$$

If, at the marginally trapped surface boundary, the result is positive (negative), and the trapped surface region lies in the direction of increasing (decreasing) χ, the region is classified as a non-cosmological trapped surface. Otherwise, the region is a cosmological trapped surface region.

3.3 Definition 3: Inner vs. outer trapped surfaces

If we seek to distinguish trapped surfaces in the general spherically symmetric dust spacetimes according to Hayward's classification scheme, then we must compute

$$L_{\mp}\theta_{\pm}|_S = \partial_{\mp}(\frac{2}{Y}[\dot{Y} - \frac{\dot{h}QY'}{t'_o \pm hQ' \pm Y'/\sqrt{1-f^2}}])|_S \tag{3.8}$$

The results of that computation are an unpublishably awful mess. Even if we rescale the null vector that we use in the definition of $\partial_{\pm}$ to remove the $\frac{2}{Y}$ scale term (since we are only interested in whether the result is positive or negative, and aren't concerned with magnitudes), the result remains impractical to actually compute, and is not obviously positive or negative from the form of the result. On the basis of pragmatics, this classification scheme seems inappropriate to employ in full generality, particularly for spacetimes with more structure than this spherically symmetric spacetime.

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References

[1] S. W. Hawking and G. F. R. Ellis, The Large-Scale Structure of Spacetime pp. 88 – 96 (Cambridge University Press, Cambridge, England, 1973).

[2] H. A. R. V. Wijesundera, "Trapped surfaces in closed Friedmann universes," Phys. Rev. D **51**, 2808 - 2812 (1995).

[3] E. Malec, and N. Ó Murchadha, "Trapped surfaces and spherical closed cosmologies," Phys. Rev. D. **47**, 1454-1464 (1993).

[4] E. Malec, and N. Ó Murchadha, "Trapped surfaces in cosmological spacetimes," Phys. Rev. D. **54**, 4799-4804 (1996).

[5] A. Qadir, J. A. Wheeler, "Late Stages of Crunch," Nuc. Phys. B (Proc. Suppl.) **6**, 345 - 348 (1989).

[6] S.A. Hayward, "General Laws of Black Hole Dynamics," Phys. Rev. D **49**, 6467 - 6474 (1994).

[7] F. J. Tipler, "Black Holes in Closed Universes," Nature **270**, 500-501 (1978).

[8] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum; E. Schmutzer, editor, *Exact Solutions of Einstein's Field Equations*, p. 160 (Cambridge University Press, Cambridge, England, 1980).

[9] D. M. Eardley and L. Smarr, "Time functions in numerical relativity: Marginally bound dust collapse" Phys. Rev. D **19**, 2239-2259 (1979).

[10] J. E. Marsden and F. J. Tipler, "Maximal Hypersurfaces and Foliations of Constant Mean Curvature in General Relativity," Phys. Reports, **66**, 109 - 139 (1980).

[11] F. J. Tipler, *The Physics of Immortality*, Doubleday, 1994, pp. 439 - 440.

[12] R. Bartnik, "Remarks on Cosmological Spacetimes and Constant Mean Curvature Surfaces," Comm. Math. Phys. **117**, 615-624 (1988).