Bernoulli on Arc Length

Victor H. Moll
Department of Mathematics, Tulane University
New Orleans, LA 70118
email: vhm@math.tulane.edu

Judith L. Nowalsky
Department of Mathematics, University of New Orleans
New Orleans, LA 70148-2900
email: jnowalsk@miltie.math.uno.edu

Gined Roa
Departmento de Matematicas, Universidad del Tolima
Ibague, Colombia
email: baruch86@latinmail.com

Leonardo Solanilla
Departmento de Matematicas, Universidad del Tolima
Ibague, Colombia
email: solanila@bunde.tolinet.com.co

Classification 01, 33

Abstract

Bernoulli’s question of the length of parabolic curves is examined and given a geometric interpretation.

1. Introduction

The academic life of the Bernoulli family was always surrounded by controversy. The disputes between Johann (John) and his older brother and former teacher Jacob and with his son Daniel are famous and well documented. An interesting discussion of this remarkable family is found in Section 12.6 of [2]. After the death of L’Hopital, John claimed the authorship of L’Hopital’s classical analysis book. In the controversy between Leibniz and Newton about the creation of Calculus, he stood on Leibniz’s side. His controversial positions were not restricted to Mathematics: he was even accused of denying the possibility of the resurrection of Christ.

In the course of our study of the history of elliptic integrals, we found a paper by Johann Bernoulli which, in our opinion, might illuminate and enrich the teaching and studying of the calculation of the arc length of smooth curves, a topic covered in most undergraduate calculus programs around the world.
The paper was written by a witty young man and contains a main theorem that is perfectly valid even today. We have found that this theorem admits a very nice interpretation in terms of the notion of curvature. We hope that teachers and students will find here a reasonable explanation of the arc length calculations that appear in the traditional Calculus texts, we also hope to provide an extra tool to invent new and interesting examples of rectifiable curves.

Let \( y = y(x) \) be a differentiable function defined on \([a, b]\). Then its arc length is defined by

\[
g(x) = \int_a^x \sqrt{1 + \left(\frac{dy}{d\xi}\right)^2} \, d\xi. \tag{1.1}
\]

In general this integral is not trivial. The examples and exercises provided in textbooks look unnatural: for instance, the first example given in Thomas [3], page 395, deals with the arclength of the curve

\[
y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \tag{1.2}
\]

for \(0 \leq x \leq 1\). This is an easy example, in the sense that the integral appearing in (1.1) is computable:

\[
g(1) = \int_0^1 \sqrt{1 + 8\xi} \, d\xi = \frac{13}{6}. \tag{1.3}
\]

The reader can verify that the integral corresponding to the length of a circle can be evaluated, but that the calculation of the arclength of an ellipse leads to the integral

\[
L(a) = a \int_0^1 \sqrt{\frac{1 - e^2 t^2}{1 - t^2}} \, dt \tag{1.4}
\]

where \(e := \sqrt{a^2 - b^2}/a\) is the eccentricity of the ellipse. The integral (1.4) is one of the fundamental elliptic integrals and is not an elementary function. It was the starting point of our research on Bernoulli’s work.

2. "Universal theorem useful for the rectification of curves"

The main result of this section is to present Bernoulli’s result on how to produce rectifiable curves.

**Theorem 2.1.** Let \( y = y(x) \) be a twice continuous differentiable function defined on the interval \([a, b]\). Define a new curve with coordinates

\[
X = x \left(\frac{dy}{dx}\right)^3 \quad \text{and} \quad Y = \frac{3x}{2} \left(\frac{dy}{dx}\right)^2 - \frac{1}{2} \int_a^x \left(\frac{dy}{d\xi}\right)^2 \, d\xi. \tag{2.1}
\]
Let \( g(x) \) and \( G(x) \) be the arc lengths of \( y \) and \( Y \) starting at \( x = a \). Then

\[
g(x) + G(x) = \left[ \xi \left( \frac{dg}{d\xi} \right)^3 \right]_{\xi=a}^{\xi=x}
\]

for all \( x \in [a,b] \).

**Proof.** First observe that

\[
g(x) = \int_a^x \sqrt{1 + \left( \frac{dy}{d\xi} \right)^2} \, d\xi
\]

so that

\[
\left( \frac{dg}{dx} \right)^3 = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}
\]

(2.4)

Following Bernoulli’s original recommendation, we compute

\[
\frac{d}{dx} x \left( \frac{dg}{dx} \right)^3 = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} + 3x \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} \frac{dy}{dx} \frac{d^2 y}{dx^2}
\]

(2.5)

On the other hand, simple differentiation shows that

\[
G(x) = \int_a^x \sqrt{1 + \left( \frac{dY}{dX} \right)^2} \, \frac{dX}{d\xi} \, d\xi
\]

\[
= \int_a^x \sqrt{1 + \left( \frac{dy}{d\xi} \right)^2} \left[ \left( \frac{dy}{d\xi} \right)^2 + 3x \frac{dy}{d\xi} \frac{d^2 y}{d\xi^2} \right] d\xi.
\]

To conclude the proof observe that the integrand of \( g(x) + G(x) \) is \( \frac{d}{dx} x \left( \frac{dg}{dx} \right)^3 \), so the result follows by the Fundamental Theorem of Calculus.

**Example.** The function \( y = \ln x \) yields

\[
X = \frac{1}{x^2} \quad \text{and} \quad Y = \frac{2}{x} - \frac{1}{2}
\]

so that

\[
Y = 2X^{1/2} - 1/2.
\]

After struggling to get the right constants in some assertions in Bernoulli’s article, we discovered a nice geometric interpretation of Theorem 2.1. This formulation eluded Bernoulli, as he did not relate his results to the *curvature* of the function \( y = y(x) \).

In the figure below we have sketched the graph of the function \( y(x) \) together with its tangent and normal lines at the point \( B \). The center of curvature is denoted by...
C and the corresponding radius of curvature is $R = BC$. Now trace the line $BM$ forming an angle $\alpha$ with the normal $BC$. The radius of curvature is defined by

$$\tan \alpha = x \frac{d^2 y}{dx^2}. \quad (2.7)$$

Now recall that in cartesian coordinates, the radius of curvature of the curve $y = y(x)$ at $x$ is

$$R(x) = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} \times \left( \frac{d^2 y}{dx^2} \right)^{-1}. \quad (2.8)$$

Thus the previous theorem can be restated in terms of the curvature.

**Theorem 2.2.** Under the assumptions of Theorem 2.1,

$$g(x) + G(x) = \left[ \xi \frac{d^2 y}{d\xi^2} R(\xi) \right]_{\xi=a}^{\xi=x}. \quad (2.9)$$

### 3. Parabolas

In old-fashioned language, a \textit{parabola} is a curve defined by the function $y = x^q$, for $q$ a rational number. In this section we discuss parabolas that are rectifiable. We remind the reader that a curve $y = y(x)$ is called rectifiable if its arclength integral admits an elementary primitive. Bernoulli was interested in the question of rectifiable parabolas and was aware of the following result.

**Theorem 3.1.** Let $n$ be a nonzero integer. Then the parabola $y = x^{2n+1/2n}$ is rectifiable on $[0,1]$.

**Proof.** The arclength is

$$g(x) = \int_0^x \sqrt{1 + \left( \frac{2n+1}{2n} \right)^2 \xi^{1/n} \ d\xi}$$

and the substitution $u(\xi) = 1 + \left( \frac{2n+1}{2n} \right)^2 \xi^{1/n}$ yields

$$g(x) = n \left( \frac{2n}{2n+1} \right)^{2n} \int_1^{u(x)} \sqrt{u} \ (u-1)^{n-1} \ du,$$

so the primitive can be found by expanding $(u-1)^{n-1}$ using the binomial theorem.

The reader will surely recognize that this result is the source of most exercises that appear in the usual textbooks. The example in (1.2) corresponds to $n = 1$. Moreover, the presence of the factor $4\sqrt{2}/3$ is not essential to the solution of the problem - it is window dressing.
We can now use Theorem 2.1 to assert that every parabola can be rectified by adding the arc length of another (conveniently chosen) parabola.

**Theorem 3.2.** The sum of the arc length integral of the (generating) parabola \( y = x^c \) and the (generated) parabola

\[
Y = \frac{3c - 2}{2c - 1} \left( \frac{x^{c/(2-3c)}}{2c-2} \right)
\]

always has a primitive. In particular, the arc length of the usual quadratic parabola (also known as the Archimedean parabola) \( y = x^2 \), is rectified by adding the arc length of the biquadratic-cubic parabola \( Y = \frac{1}{3} 2^{7/4} X^{3/4} \).

**Proof.** Let \( y = x^c \), then \( X = c^3 x^{3c-2} \) and

\[
Y = \frac{3}{2} c^2 x^{2c-1} - \frac{1}{2} c^2 \int_0^x \xi^{2c-2} \ d\xi = \frac{c^2 (3c-2)}{2c-1} x^{2c-1}
\]

\[
= \frac{3c - 2}{2c - 1} e^{(6c-7)/(3c-2)} X^{(2c-1)/(3c-2)}.
\]

Many interesting question can be formulated at this point. For instance, when is the degree of the generating and generated parabolas the same? (Hint: the non-trivial answer to this question was known in Bernoulli’s time as the primary cubic parabola.)

We have shown that parabolas of the form \( y = x^{2n+1} \) are rectifiable. Is it possible that both the generating and generated parabolas are of this type?

**References**

