SUMS OF ARCTANGENTS AND SOME FORMULAE OF RAMANUJAN

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Abstract. We present diverse methods to evaluate arctangent and related sums.

1. Introduction

The evaluation of arctangent sums of the form

\[ \sum_{k=1}^{\infty} \tan^{-1} h(k) \]  

for a rational function \( h \) reappear in the literature from time to time. For instance the evaluation of

\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} = \frac{3\pi}{4} \]  

is completely elementary and was proposed by J. Anglesio [1] in 1993. This is a classical problem that appears in [7, 9, 13], among other places. The evaluation of

\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2} = \tan^{-1} \left( \frac{\tan(\pi/\sqrt{2}) - \tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2}) + \tanh(\pi/\sqrt{2})} \right) \]  

is somewhat less elementary. R.J. Chapman [6] proposed in 1990 to evaluate this sum in closed form (in terms of a finite number of trigonometric and hyperbolic functions). This was solved by A. Sarkar [15] using the techniques described in Section 3.

The goal of this paper is to discuss the evaluation of these sums and the related rational sums

\[ S(n) = \sum_{k=1}^{n} R(k) \]  

for a rational function \( R \). Note that \( \tan^{-1} x \) will always denote the principal value.

We make use of the addition formula for \( \tan^{-1} x \):

\[ \tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \frac{x+y}{1-xy} & \text{if } xy < 1 \\ \tan^{-1} \frac{x+y}{1-xy} + \pi \text{ sign } x & \text{if } xy > 1 \end{cases} \]

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and

\[(1.6) \quad \tan^{-1}x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2} \text{ sign } x.\]

2. The method of telescoping

The closed-form evaluation of a finite sum

\[S(n) := \sum_{k=1}^{n} a_k\]

is elementary if one can find a sequence \(\{b_k\}\) such that

\[a_k = b_k - b_{k-1}.\]

In this situation, the sum \(S(n)\) telescopes, i.e.

\[S(n) := \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k - b_{k-1} = b_n - b_0.\]

This method can be extended to situations in which the telescoping nature of \(a_k\) is hidden by a function.

**Theorem 2.1.** Let \(f(x)\) be of fixed sign and define \(h\) by

\[(2.1) \quad h(x) = \frac{f(x+1) - f(x)}{1 + f(x+1)f(x)}.\]

Then

\[(2.2) \quad \sum_{k=1}^{n} \tan^{-1}h(k) = \tan^{-1}f(n+1) - \tan^{-1}f(1).\]

In particular, if \(f\) has a limit at \(\infty\) (including the possibility of \(f(\infty) = \infty\)), then

\[(2.3) \quad \sum_{k=1}^{\infty} \tan^{-1}h(k) = \tan^{-1}f(\infty) - \tan^{-1}f(1).\]

**Proof.** Since

\[\tan^{-1}h(k) = \tan^{-1}f(k+1) - \tan^{-1}f(k),\]

(2.2) follows by telescoping. \(\square\)

**Note.** The hypothesis on the sign of \(f\) is included in order to avoid the case \(xy > 1\) in (1.5). In general, (2.2) has to be replaced by

\[(2.4) \quad \sum_{k=1}^{n} \tan^{-1}h(k) = \tan^{-1}f(n) - \tan^{-1}f(1) + \pi \sum \text{ sign } f(k),\]

where the sum is taken over all \(k\) between 1 and \(n\) for which \(f(k)f(k+1) < -1\). Thus (2.2) is always correct up to an integral multiple of \(\pi\). The restrictions on the parameters in the examples described in this paper have the intent of keeping \(f(k), k \in \mathbb{N}\) of fixed sign.

**Example 2.1.** Let \(f(x) = ax + b\), where \(a, b\) are such that \(f(x) \geq 0\) for \(x \geq 1\). Then

\[(2.5) \quad h(x) = \frac{a}{a^2x^2 + a(a+2b)x + (1+ab+b^2)},\]
and (2.3) yields

\[
\sum_{k=1}^{\infty} \frac{1}{a^2k^2 + a(a + 2b)k + (1 + ab + b^2)} = \frac{\pi}{2} - \frac{1}{\tan^{-1}(a + b)}.
\]

**Special cases:** \(a = 1\) and \(b = 0\) give \(f(x) = x\) and \(h(x) = 1/(x^2 + x + 1)\), and the sum is

\[
\sum_{k=1}^{\infty} \frac{1}{k^2 + k + 1} = \frac{\pi}{4}.
\]

- \(a = 2\) and \(b = 0\) give \(f(x) = 2x\) and \(h(x) = 2/(2x + 1)^2\), so that

\[
\sum_{k=0}^{\infty} \frac{2}{(2k + 1)^2} = \frac{\pi}{2}.
\]

Differentiating (2.6) with respect to \(a\) yields

\[
\sum_{k=1}^{\infty} \frac{p_{a,b}(k)}{q_{a,b}(k)} = \frac{1}{1 + (a + b)^2},
\]

where

\[
p_{a,b}(k) = a^2k^2 + a^2k - (1 + b^2)
\]

and

\[
q_{a,b}(k) = a^4k^4 + 2a^3(a + 2b)k^3 + a^2(2 + a^2 + 6ab + 6b^2)k^2 + 2a(a + 2b)(1 + ab + b^2)k + (1 + b^2)(1 + a^2 + 2ab + b^2).
\]

**Special cases:** \(a = 1\) and \(b = 0\) give

\[
\sum_{k=1}^{\infty} \frac{k^2 + k - 1}{k^4 + 2k^3 + 3k^2 + 2k + 2} = \frac{1}{2}.
\]

- \(a = 1/2\) and \(b = 1/3\) give

\[
\sum_{k=1}^{\infty} \frac{9k^2 + 9k - 40}{81k^4 + 378k^3 + 1269k^2 + 1932k + 2440} = \frac{1}{61}.
\]

- Let \(a = -2b\) and differentiate with respect to \(b\) to produce

\[
\sum_{k=1}^{\infty} \frac{4b^2k^2 - (1 + b^2)}{16b^4k^4 + 8b^2(1 - b^2)k^2 + (1 + b^2)^2} = \frac{1}{2(1 + b^2)}.
\]

Mathematica evaluates this last sum.

**Example 2.2.** This example considers the quadratic function \(f(x) = ax^2 + bx + c\) under the assumption that \(f(k)\), \(k \in \mathbb{N}\) has fixed sign. For instance this happens if \(b^2 - 4ac \leq 0\). Define

\[
a_0 := 1 + ac + bc + c^2 \\
a_1 := ab + b^2 + 2ac + 2bc \\
a_2 := a^2 + 3ab + b^2 + 2ac \\
a_3 := 2a(a + b) \\
a_4 := a^2
\]
Then
\[ h(x) = \frac{2ax + a + b}{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0} \]
and thus
\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{2ak + a + b}{a_4 k^4 + a_3 k^3 + a_2 k^2 + a_1 k + a_0} = \frac{\pi}{2} - \tan^{-1}(a + b + c). \]

**Special cases:** \( b = -a, \ c = a/2 \) yields
\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{8ak}{4a^2 k^4 + (a^2 + 4)} = \frac{\pi}{2} - \tan^{-1}\frac{a}{2}. \]

The particular case \( a = 1, \ b = -1 \) and \( c = 1/2 \) gives
\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{2ak}{a^2 k^4 - a^2 k^2 + 1} = \frac{\pi}{2}, \]
which is independent of \( a \).

Additional examples can be given by telescoping twice (or even more).

**Corollary 2.2.** Let \( f \) and \( h \) be related by
\[ h(x) = \frac{f(x + 1) - f(x - 1)}{1 + f(x + 1) f(x - 1)}. \]

Then
\[ \sum_{k=1}^{n} \tan^{-1} h(k) = \tan^{-1} f(n + 1) - \tan^{-1} f(1) + \tan^{-1} f(n) - \tan^{-1} f(0). \]

In particular,
\[ \sum_{k=1}^{\infty} \tan^{-1} h(k) = 2 \tan^{-1} f(\infty) - \tan^{-1} f(1) - \tan^{-1} f(0). \]

**Proof.** The relation (2.13) shows that
\[ \tan^{-1} h(k) = \tan^{-1} f(k + 1) - \tan^{-1} f(k - 1). \]

Thus
\[ \sum_{k=1}^{n} \tan^{-1} h(k) = \sum_{k=1}^{n} [\tan^{-1} f(k + 1) - \tan^{-1} f(k - 1)] \]
\[ = \sum_{k=1}^{n} [\tan^{-1} f(k + 1) - \tan^{-1} f(k)] + \sum_{k=1}^{n} [\tan^{-1} f(k) - \tan^{-1} f(k - 1)] \]
\[ = \tan^{-1} f(n + 1) - \tan^{-1} f(1) + \tan^{-1} f(n) - \tan^{-1} f(0). \]

\[ \square \]
Example 2.3. The evaluation
\[
\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} = \frac{3\pi}{4}
\]
corresponds to \(f(k) = k\) so that \(h(k) = 2/k^2\). This is the problem proposed by Anglesio [1].

Example 2.4. Take \(f(k) = -2/k^2\) so that \(h(k) = 8k/(k^4 - 2k^2 + 5)\). It follows that
\[
\sum_{k=1}^{\infty} \tan^{-1} \frac{8k}{k^4 - 2k^2 + 5} = \pi - \tan^{-1} \frac{1}{2}.
\]
This sum is part b) of the Problem proposed in [1].

Example 2.5. Take \(f(k) = -a/(k^2 + 1)\). Then \(h(k) = 4ak/(k^4 + a^2 + 4)\), so that
\[
\sum_{k=1}^{\infty} \tan^{-1} \frac{4ak}{k^4 + a^2 + 4} = \tan^{-1} \frac{a}{2} + \tan^{-1} a.
\]

Special cases: \(a = 1\) yields
\[
\sum_{k=1}^{\infty} \tan^{-1} \frac{4k}{k^4 + 5} = \frac{\pi}{4} + \tan^{-1} \frac{1}{2}.
\]

\(\bullet\) \(a = \sqrt{2}\) gives
\[
\sum_{k=1}^{\infty} \tan^{-1} \frac{4\sqrt{2}k}{k^4 + 6} = \frac{\pi}{2}.
\]

\(\bullet\) Differentiating (2.16) with respect to \(a\) gives
\[
\sum_{k=1}^{\infty} \frac{4k(k^4 + 4 - a^2)}{k^8 + 2(a^2 + 4)k^4 + 16a^2k^2 + (a^4 + 8a^2 + 16)} = \frac{3(a^2 + 2)}{(a^2 + 1)(a^2 + 4)}.
\]
The special case \(a = 0\) yields
\[
\sum_{k=1}^{\infty} \frac{k}{k^4 + 4} = \frac{3}{8}
\]
and \(a = 2\) gives
\[
\sum_{k=1}^{\infty} \frac{k^5}{k^8 + 16k^4 + 64k^2 + 64} = \frac{9}{80}.
\]
Mathematica evaluates the first sum but not the second.

The examples described above are rather artificial. The interesting question is to find \(f(x)\) given the function \(h\). In general it is not possible to find \(f\) in closed form. In the case of Example 2.3, we need to solve the functional equation
\[
2 [1 + f(x - 1)f(x + 1)] = x^2 [f(x + 1) - f(x - 1)].
\]
A polynomial solution of (2.21) must have degree at most 2 and trying \(f(x) = ax^2 + bx + c\) yields \(f(x) = x\).
3. The method of zeros

A different technique for the evaluation of arctangent sums is based on the factorization of the product

\[ p_n := \prod_{k=1}^{n} (a_k + ib_k) \]

with \( a_k, b_k \in \mathbb{R} \). The argument of \( p_n \) is given by

\[ \text{Arg}(p_n) = \sum_{k=1}^{n} \tan^{-1} \frac{b_k}{a_k} \]

**Example 3.1.** Let

\[ p_n(z) = \prod_{k=1}^{n} (z - z_k) \]

be a polynomial with real coefficients. Then

\[ \text{Arg}(p_n(z)) = \sum_{k=1}^{n} \tan^{-1} \frac{x - x_k}{y - y_k}. \]

The special case \( p_n(z) = z^n - 1 \) has roots at \( z_k = \cos(2\pi k/n) + i \sin(2\pi k/n) \), so we obtain

\[ \text{Arg}(z^n - 1) = \sum_{k=1}^{n} \tan^{-1} \frac{x - \cos(2\pi k/n)}{y - \sin(2\pi k/n)}. \]

**Example 3.2.** The classical factorization

\[ \sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) \]

yields the evaluation

\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{2xy}{k^2 - x^2 + y^2} = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{\tanh \pi y}{\tan \pi x}. \]

**Special cases:** \( x = y \) yields

\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{2x^2}{k^2} = \frac{\pi}{4} - \tan^{-1} \frac{\tanh \pi x}{\tan \pi x}. \]

- \( x = y = 1/\sqrt{2} \) gives

\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2} = \frac{\pi}{4} - \tan^{-1} \frac{\tanh (\pi/\sqrt{2})}{\tan(\pi/\sqrt{2})}, \]

which corresponds to (1.3).

- \( x = y = 1/2 \) yields

\[ \sum_{k=1}^{\infty} \tan^{-1} \frac{1}{2k^2} = \frac{\pi}{4} \]
Differentiating (3.7) gives

\[
\sum_{k=1}^{\infty} \frac{k^2}{k^4 + 4x^4} = \frac{\pi}{4x} \cos 2\pi x - \cosh 2\pi x.
\]

In particular, \(x = 1\) yields

\[
\sum_{k=1}^{\infty} \frac{k^2}{k^4 + 4} = \frac{\pi}{4} \coth \pi.
\]

The identity (3.10) is comparable to Ramanujan’s evaluation

\[
\sum_{k=1}^{\infty} \frac{k^2}{k^4 + x^2k^2 + x^4} = \frac{\pi}{2x\sqrt{3}} \sinh \pi x \sqrt{3} - \sqrt{3} \sin \pi x \cosh \pi x \sqrt{3} - \cos \pi x
\]

discussed in [3], Entry 4 of Chapter 14.

Glasser and Klamkin [10] present other examples of this technique.

4. A FUNCTIONAL EQUATION

The table of sums and integrals [11] contains a small number of examples of finite sums that involve trigonometric functions of multiple angles. In Section 1.36 we find

\[
\sum_{k=1}^{n} 2^{2k} \sin^4 \frac{x}{2k} = 2^n \sin^2 \frac{x}{2^n} - \sin^2 x
\]

and Section 1.37 consists entirely of the two sums

\[
\sum_{k=0}^{n} \left( \frac{1}{2k} \right)^n \tan \frac{x}{2k} = \frac{1}{2^n} \cot \frac{x}{2^n} - 2 \cot 2x
\]

\[
\sum_{k=0}^{n} \left( \frac{1}{2k} \right)^n \tan^2 \frac{x}{2k} = \frac{2^{2n+2} - 1}{3 \cdot 2^n - 1} + 4 \cot^2 2x - \frac{1}{2^n} \cot^2 \frac{x}{2^n}.
\]

In this section we present a systematic procedure to analyze these sums.

**Theorem 4.1.** Let

\[
F(x) = \sum_{k=1}^{\infty} f(x, k) \quad \text{and} \quad G(x) = \sum_{k=1}^{\infty} (-1)^k f(x, k).
\]

Suppose \(f(x, 2k) = \nu f(\lambda(x), k)\) for some \(\nu \in \mathbb{R}\) and a function \(\lambda : \mathbb{R} \to \mathbb{R}\). Then

\[
F(x) = (2\nu)^n F(\lambda^{(n)}(x)) - \sum_{j=0}^{n-1} (2\nu)^j G(\lambda^{(j)}(x)).
\]

**Proof.** Observe that

\[
F(x) + G(x) = 2 \sum_{k=1}^{\infty} f(x, 2k) = 2\nu \sum_{k=1}^{\infty} f(\lambda(x), k) = 2\nu F(\lambda(x)).
\]

Repeat this argument to obtain the result.
Example 4.1. Let \( f(x,k) = 1/(x^2 + k^2) \), so that \( \nu = 1/4 \) and \( \lambda(x) = x/2 \). Since
\[
F(x) = \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} = \frac{\pi x \coth \pi x - 1}{2x^2},
\]
and
\[
G(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 + k^2} = \frac{\pi x \text{csch} \pi x - 1}{2x^2},
\]
(4.6) yields, upon letting \( n \to \infty \),
\[
\sum_{j=0}^{\infty} \frac{x}{\sinh 2^{-j}x} - 2^j = 1 - \frac{x}{\tanh x}.
\]
Now replace \( x \) by \( \ln t \), differentiate with respect to \( t \), and set \( t = e \) to produce
\[
\sum_{j=0}^{\infty} \frac{2^j - \coth 2^{-j}}{2^j \sinh 2^{-j}} = \frac{1 + 4e^2 - e^4}{1 - 2e^2 + e^4}.
\]
Now go back to (4.7), replace \( x \) by \( \ln t \), differentiate with respect to \( t \), set \( t = ae \), differentiate with respect to \( a \), and set \( a = e \) to produce
\[
\sum_{j=0}^{\infty} \frac{2 - 2^2j + \text{csch}^2 2^{-j} - \text{sech}^2 2^{-j}}{2^j \sinh 2^1 - 2^{-j}} = \frac{e^{12} - 17e^8 - 17e^4 + 1}{e^{12} - 3e^8 + 3e^4 - 1}.
\]

Corollary 4.2. Let
\[
F(x) = \sum_{k=1}^{\infty} f \left( \frac{x}{k} \right) \quad \text{and} \quad G(x) = \sum_{k=1}^{\infty} (-1)^k f \left( \frac{x}{k} \right).
\]
Then, for any \( n \in \mathbb{N} \),
\[
F(x) = 2^{-n} F(2^n x) + \sum_{k=1}^{n} 2^{-k} G(2^k x).
\]
In particular, if \( F \) is bounded, then
\[
F(x) = \sum_{k=1}^{\infty} 2^{-k} G(2^k x).
\]
Proof. The function \( f(x/k) \) satisfies the conditions of Theorem 4.1 with \( \nu = 1 \) and \( \lambda(x) = x/2 \). Thus
\[
F(x) = 2^n F(x/2^n) - \sum_{j=1}^{n-1} 2^j G(x/2^j).
\]
Now replace \( x \) by \( x/2^n \) to obtain (4.11). Finally let \( n \to \infty \) to obtain (4.12). 

The key to the proof of Theorem 4.1 is the identity \( F(x) + G(x) = 2\nu F(\lambda(x)) \).

We now present an extension of this result.
Theorem 4.3. Let $F$, $G$ be functions that satisfy
\begin{equation}
F(x) = r_1 F(m_1 x) + r_2 G(m_2 x)
\end{equation}
for some parameters $r_1, r_2, m_1, m_2$. Then
\begin{equation}
\sum_{k=1}^{n} r_1^{k-1} G(m_1^{k-1} m_2 x) = F(x) - r_1^n F(m_1^n x).
\end{equation}

Proof. Replace $x$ by $m_1 x$ in (4.13) to produce
\begin{equation}
F(m_1 x) = r_1 F(m_1^2 x) + r_2 G(m_1^2 m_2 x),
\end{equation}
which, when combined with (4.13), gives
\begin{equation}
F(x) = r_1^2 F(m_1^2 x) + r_1 r_2 G(m_1 m_2 x) + r_2 G(m_2 x).
\end{equation}
Formula (4.14) follows by induction. \qed

We now present two examples that illustrate Theorem 4.3. These sums appear as entries in Ramanujan’s Notebooks.

Example 4.2. The identity
\begin{equation}
cot x = \frac{1}{2} \cot \frac{x}{2} - \frac{1}{2} \tan \frac{x}{2}
\end{equation}
shows that $F(x) = \cot x$, $G(x) = \tan x$ satisfy (4.13) with $r_1 = 1/2$, $r_2 = -1/2$ and $m_1 = m_2 = 1/2$. We conclude that
\begin{equation}
\sum_{k=1}^{n} 2^{-k} \tan \frac{x}{2^k} = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x.
\end{equation}
This is (4.3). It also appears as Entry 24, p. 364, of Ramanujan’s Third Notebook as described in Berndt [4], p. 396. Similarly, the identity
\begin{equation}
\sin^2(2x) = 4 \sin^2 x - 4 \sin^4 x
\end{equation}
yields (4.1). The reader is invited to produce proofs of (4.2) and (4.4) in the style presented here.

Example 4.3. The identity
\begin{equation}
cot x = \cot \frac{x}{2} - \csc x
\end{equation}
satisfies (4.13) with $F(x) = \cot x$, $G(x) = \csc x$ and parameters $r_1 = 1$, $r_2 = -1$, $m_1 = 1/2$, $m_2 = 1$. We obtain
\begin{equation}
\sum_{k=1}^{n} \csc \frac{x}{2^{k-1}} = \cot \frac{x}{2^n} - \cot x.
\end{equation}
This appears in the proof of Entry 27 of Ramanujan’s Third Notebook in Berndt [4], p. 398.
Example 4.4. The application of Theorem 4.1, or Corollary 4.2, requires an analytic expression for \( F \) and \( G \). A source of such expressions is Jolley [12]. Indeed, entries 578 and 579 are

\[
\sum_{k=1}^{\infty} \frac{\tan^{-1} \frac{2x^2}{k^2}}{\tan \pi x} = \frac{\pi}{4} - \frac{\tan^{-1} \frac{x}{\tan \pi x} \tanh \pi x}{\pi x}.
\]

These results also appear in [5], page 314. Applying one step of Proposition 4.2 we conclude that

\[
2 \tan^{-1} \frac{\tan x}{\tan 2x} = \tan^{-1} \frac{2x}{\sin 2x} + \tan^{-1} \frac{2x}{\sin 2x}.
\]

We also obtain

\[
\sum_{k=1}^{n} 2^{-k} \tan^{-1} \frac{\sin 2^k x}{\sin 2^k x} = \tan^{-1} \frac{x}{\tan x} - 2^{-n} \tan^{-1} \frac{\tan 2^n x}{\tan 2^n x},
\]

and by the boundedness of \( \tan^{-1} x \) conclude that

\[
\sum_{k=1}^{\infty} 2^{-k} \tan^{-1} \frac{\sin 2^k x}{\sin 2^k x} = \tan^{-1} \frac{x}{\tan x}.
\]

Differentiating (4.23) gives

\[
2 \sum_{k=1}^{n} \frac{\cos 2^k x \sin 2^k x - \cosh 2^k x \sin 2^k x}{\cos 2^k x - \cosh 2^k x} = \frac{\sin 2x - \sin 2x}{\cos 2x - \cosh 2x} + \frac{\sin 2^{n+1} x - \sin 2^{n+1} x}{\cos 2^{n+1} x - \cosh 2^{n+1} x}.
\]

Letting \( n \to \infty \) and using the identity

\[
\cos 2^{k+1} x - \cosh 2^{k+1} x = -2 (\sin^2 2^k x + \sinh^2 2^k x)
\]

yields

\[
\sum_{k=1}^{\infty} \frac{\cosh 2^k x \sin 2^k x - \sinh 2^k x \cos 2^k x}{\sin^2 2^k x + \sinh^2 2^k x} = \frac{\sec^2 x \tan x - \tanh x \sec^2 x}{\tan^2 x + \tanh^2 x} + \text{sign } x.
\]

For example, \( x = \pi \) gives

\[
\sum_{k=1}^{\infty} \frac{1}{\sinh 2^k \pi} = \coth \pi - 1.
\]

5. Reduction to telescoping

The sum

\[
S(f) = \sum_{k=1}^{n} f(k, t),
\]

depending on the parameter \( t \), is said to telescope if the summand can be written in the form

\[
f(k, t) = f_1(k, t) - f_1(k + m, t)
\]
for a function \(f_1\) and \(m \in \mathbb{Z}\) fixed. In this section we discuss a method to determine if the sum \(S\) telescopes.

We first consider the case of a function \(f(k, t)\) that is rational in \(k\). Then the question of telescoping is decided by examining the partial fraction decomposition of \(f\). For simplicity, we assume that the poles of \(f\) are simple and we omit the parameter \(t\).

**Proposition 5.1.** Let
\[
f(k) = \sum_{j=1}^{r} \frac{a_j}{k - k_j}
\]
be the partial fraction decomposition of \(f\). Then \(S(f)\) telescopes if and only if \(f\) can be written as
\[
f(k) = \sum_{j=1}^{s} \left( \frac{b_j}{k - k_j} - \frac{b_j}{k - k_j - m_j} \right)
\]
where \(m_j \in \mathbb{Z}\).

We now observe that if (5.1) telescopes, then so do the sums
\[
\sum_{k=1}^{n} \frac{\partial}{\partial k} f(k, t) \quad \text{and} \quad \sum_{k=1}^{n} \frac{\partial}{\partial t} f(k, t).
\]
In particular, the question of telescoping of the arctangent sum
\[
S(n) = \sum_{k=1}^{n} \tan^{-1} R(k)
\]
for a rational function \(R\) is reduced to that of the rational sum
\[
S_1(n) = \sum_{k=1}^{n} \frac{1}{1 + R^2(k)} \frac{\partial R}{\partial k},
\]
by differentiating with respect to \(k\). For instance, the sum in (2.12) produces a rational function with poles at
\[
\frac{1}{2}(\pm 1 - \sqrt{1 - 4i/a}), \quad \frac{1}{2}(\pm 1 + \sqrt{1 - 4i/a}), \quad \frac{1}{2}(\pm 1 - \sqrt{1 + 4i/a}), \quad \frac{1}{2}(\pm 1 + \sqrt{1 + 4i/a}),
\]
and these can be paired as required in Proposition 5.1.

**Example 5.1.** Consider the sum
\[
S = \sum_{k=1}^{\infty} \left[ k \tan^{-1} \frac{2k^2 - 2k + 1}{k^4 - 2k^3 + 2k^2 - 1} - (k^2 + 1)\tan^{-1} \frac{2k - 1}{k^4 - 2k^3 + 2k^2 + 1} \right].
\]
To evaluate \(S\) we introduce
\[
A(k) = k \tan^{-1} \frac{2k^2 - 2k + 1}{k^4 - 2k^3 + 2k^2 - 1} - (k^2 + 1)\tan^{-1} \frac{2k - 1}{k^4 - 2k^3 + 2k^2 + 1}
\]
and observe that
\[
\frac{d^3}{dk^3} A(k) = B(k) - B(k-1),
\]
where
\[
B(k) = -\frac{6}{k^4 + 1} - \frac{8(3k^3 - 5k - 8)}{(k^4 + 1)^2} + \frac{64(k^3 - k - 1)}{(k^4 + 1)^3}.
\]
Let $C(k)$ be the function obtained by integrating $B(k)$ three times:

\[
C(k) = -(k^2 + k + 1)\tan^{-1}k^2 + P_2(k)
\]

where $P_2(k)$ is a polynomial of degree 2.

Integrating backwards we obtain

\[
\sum_{k=2}^{N} \left[ k \tan^{-1} \frac{2k^2 - 2k + 1}{k^4 - 2k^3 + k^2 - 1} - (k^2 + 1) \tan^{-1} \frac{2k - 1}{k^4 - 2k^3 + k^2 + 1} \right] = C(N) - C(1)
\]

\[
= -(N^2 + N + 1)\tan^{-1}N^2 + Q_2(N),
\]

where $Q_2(N) = P_2(N) - C(1)$ is another polynomial of degree 2. We have started the sum at $k = 2$ because $k^4 - 2k^3 + k^2 - 1$ has a zero at $k = \frac{1}{2} (\sqrt{5} + 1) \sim 1.618$.

The polynomial $Q_2(N) = aN^2 + bN + c$ can be determined from

\[
aN^2 + bN + c = \sum_{k=2}^{N} A(k) + (N^2 + N + 1)\tan^{-1}N^2
\]

by evaluating (5.11) at three distinct values of $N$. The result is

\[
Q_2(N) = \frac{\pi}{4} (2N^2 + 2N - 1).
\]

Thus

\[
\sum_{k=2}^{N} \left[ k \tan^{-1} \frac{2k^2 - 2k + 1}{k^4 - 2k^3 + k^2 - 1} - (k^2 + 1) \tan^{-1} \frac{2k - 1}{k^4 - 2k^3 + k^2 + 1} \right] = -(N^2 + N + 1)\tan^{-1}N^2 + \frac{\pi}{4} (2N^2 + 2N - 1),
\]

so that

\[
\sum_{k=1}^{\infty} \left[ k \tan^{-1} \frac{2k^2 - 2k + 1}{k^4 - 2k^3 + k^2 - 1} - (k^2 + 1) \tan^{-1} \frac{2k - 1}{k^4 - 2k^3 + k^2 + 1} \right] = 1 - \frac{3\pi}{2}.
\]

**Note.** Naturally this idea applies to a more general class of sums. For instance, sums of the form

\[
L(n) = \sum_{k=1}^{n} p(k) \ln R(k)
\]

and

\[
A(n) = \sum_{k=1}^{n} p(k) \tan^{-1}R(k),
\]

for a polynomial $p$, can be reduced to a sum with rational summand by successive differentiation.
6. A DYNAMICAL SYSTEM

In this section we describe a dynamical system that appears in the evaluation of arctangent sums. Define

\[ x_n = \tan \sum_{k=1}^{n} \tan^{-1} k \quad \text{and} \quad y_n = \tan \sum_{k=1}^{n} \tan^{-1} \frac{1}{k}. \]

Then \( x_1 = y_1 = 1 \) and

\[ x_n = \frac{x_{n-1} + n}{1 - nx_{n-1}} \quad \text{and} \quad y_n = \frac{ny_{n-1} + 1}{n - y_{n-1}}. \]

**Proposition 6.1.** Let \( n \in \mathbb{N} \). Then

\[ x_n = \begin{cases} -y_n & \text{if } n \text{ is even} \\ 1/y_n & \text{if } n \text{ is odd} \end{cases} \]

that is

\[ \tan \sum_{k=1}^{n} \tan^{-1} k = -\tan \sum_{k=1}^{n} \tan^{-1} \frac{1}{k} \]

if \( n \) is even and

\[ \tan \sum_{k=1}^{n} \tan^{-1} k = \cotg \sum_{k=1}^{n} \tan^{-1} \frac{1}{k} \]

if \( n \) is odd.

**Proof.** The recurrence formulas for \( x_n \) and \( y_n \) can be used to prove the result directly. A pure trigonometric proof is presented next. If \( n \) is even then

\[
\tan \sum_{k=1}^{2m} \tan^{-1} k + \tan \sum_{k=1}^{2m} \tan^{-1} \frac{1}{k} = \tan \sum_{k=1}^{2m} \tan^{-1} k + \tan \sum_{k=1}^{2m} \left( \frac{\pi}{2} - \tan^{-1} k \right)
\]

\[
= \tan \sum_{k=1}^{2m} \tan^{-1} k + \tan \left( \pi m - \sum_{k=1}^{2m} \tan^{-1} k \right)
\]

\[
= 0.
\]

A similar argument holds for \( n \) odd.

This dynamical system suggests many interesting questions. We conclude by proposing one of them: **Observe that** \( x_3 = 0. \) **Does this ever happen again?**

**References**


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