PROOF OF FORMULA 4.224.7

\[ \int_0^{\pi/2} \ln^2 \sin x \, dx = \frac{\pi}{2} \left[ \ln^2 2 + \frac{\pi^2}{12} \right] \]

The recurrence presented here is due to M. G. Beumer, Amer. Math. Monthly 68(1961), 645-647. Let

\[ S_n := \int_0^{\pi/2} \ln^n \sin x \, dx \]

and define the Dirichlet series

\[ X(s) = \sum_{j=0}^{\infty} \frac{2^{-2j} \binom{2j}{j}}{(2j+1)^s}. \]

Using

\[ \int_0^\infty x^{s-1} e^{-(2n+1)x} \, dx = \frac{\Gamma(s)}{(2n+1)^s}, \]

the binomial theorem

\[ (1-t)^{-1/2} = \sum_{k=0}^{\infty} 2^{-2k} \binom{2k}{k} t^k \]

yields

\[ X(s) \Gamma(s) = \int_0^\infty \frac{x^{s-1} \, dx}{\sqrt{e^{2x} - 1}}. \]

The change of variables \( e^{-x} = \sin \theta \) shows that \( 2(n-1)! X_n = (-1)^{n-1} S_{n-1} \).

Integrating the relation

\[ (\sin t)^x = \sum_{n=0}^{\infty} \ln^n \sin t \frac{x^n}{n!} \]

produces

\[ \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} S_n = \int_0^{\pi/2} \sin^x t \, dt = \frac{\sqrt{\pi} \Gamma(x/2 + 1/2)}{2 \Gamma(x/2 + 1)}. \]

Differentiate logarithmically and use

\[ \psi \left( \frac{x}{2} + \frac{1}{2} \right) - \psi \left( \frac{x}{2} + 1 \right) = -2 \ln 2 + 2 \sum_{n=1}^{\infty} (-1)^{n+1} (1 - 2^{-n}) \zeta(n+1) x^n, \]

yields, after matching coefficients, the recurrence

\[ S_n = -\ln 2 \ S_{n-1} - \sum_{r=1}^{n-1} (-1)^r (1 - 2^{-r}) \frac{(n-1)! \zeta(r+1)}{(n-1-r)!} \ S_{n-1-r}. \]

The initial value \( S_0 = \pi \) now gives all the values \( S_n \). In particular, \( S_1 = -\pi \ln 2 \) produces the stated value of \( S_2 \).