AN EXTENSION OF A CRITERION FOR UNIMODALITY

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Abstract. We prove that if \( P(x) \) is a polynomial with nonnegative nondecreasing coefficients and \( n \) is a positive integer, then \( P(x + n) \) is unimodal. Applications and open problems are presented.

1. Introduction

A finite sequence of real numbers \( \{d_0, d_1, \ldots, d_m\} \) is said to be unimodal if there exists an index \( 0 \leq m^* \leq m \), called the mode of the sequence, such that \( d_j \) increases up to \( j = m^* \) and decreases from then on, that is, \( d_0 \leq d_1 \leq \cdots \leq d_{m^*} \) and \( d_{m^*} \geq d_{m^*+1} \geq \cdots \geq d_m \). A polynomial is said to be unimodal if its sequence of coefficients is unimodal.

Unimodal polynomials arise often in combinatorics, geometry and algebra. The reader is referred to [3] and [4] for surveys of the diverse techniques employed to prove that specific families of polynomials are unimodal.

A sequence of positive real numbers \( \{d_0, d_1, \ldots, d_m\} \) is said to be logarithmic concave (or log concave for short) if \( d_{j+1}d_{j-1} \leq d_{j+1}^2 \) for \( 1 \leq j \leq m-1 \). It is easy to see that if a sequence is log concave then it is unimodal [5]. A sufficient condition for log concavity of a polynomial is given by the location of its zeros: if all the zeros of a polynomial are real and negative, then it is log concave and therefore unimodal [5]. A simple criterion for unimodality was established in [2]: if \( a_j \) is a nondecreasing sequence of positive real numbers, then

\[
P(x + 1) = \sum_{j=0}^{m} a_j(x + 1)^j
\]

for \( j = 0 \) to \( m \),

\[
P(x) = \sum_{j=0}^{m} d_j(m)x^j
\]

is unimodal. This criterion is reminiscent of Brenti’s criterion for log concavity [3]. A sequence of real numbers is said to have no internal zeros if \( d_i, d_k \neq 0 \) and \( i < j < k \) imply \( d_j \neq 0 \). Brenti’s criterion states that if \( P(x) \) is a log concave polynomial with nonnegative coefficients and with no internal zeros, then \( P(x + 1) \)

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is log concave.

In this paper we first prove that under the same conditions of [2] the polynomial $P(x + n)$ is unimodal for any $n \in \mathbb{N}$, the set of positive integers. We also characterize the unimodal sequences $\{d_j\}$ that appear in [2] and discuss the behavior of the coefficients of $P(x + 1)$ for a unimodal polynomial $P(x)$. Numerical evidence suggests that the unimodality result is true for $n$ real and positive. This remains to be investigated.

2. The extension

In this section we prove an extension of the main result in [2]. We start by establishing an elementary inequality.

**Lemma 2.1.** Let $m, n \in \mathbb{N}$ and $m_* := \lfloor \frac{m}{n+1} \rfloor$. Then $(n+1)m_* \leq m \leq (n+1)m_* + n$.

**Proof.** This follows directly from $\frac{m}{n+1} - 1 < m_* \leq \frac{m}{n+1}$. \hfill \Box

**Theorem 2.2.** Let $0 \leq a_0 \leq a_1 \cdots \leq a_m$ be a sequence of real numbers and $n \in \mathbb{N}$, and consider the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m. \tag{2.1}$$

Then the polynomial $P(x + n)$ is unimodal with mode $m_* = \lfloor \frac{m}{n+1} \rfloor$.

We now restate Theorem 2.2 in terms of the coefficients of $P$.

**Theorem 2.3.** Let $0 \leq a_0 \leq a_1 \cdots \leq a_m$ be a sequence of real numbers and $n \in \mathbb{N}$. Then the sequence

$$q_j := q_j(m, n) = \sum_{k=j}^{m} a_k \binom{k}{j} n^{k-j} \tag{2.2}$$

is unimodal with mode $m_* = \lfloor \frac{m}{n+1} \rfloor$.

**Proof.** The coefficients $q_j(m)$ in (1.2) are given by

$$q_j(m) = \sum_{k=j}^{m} a_k \binom{k}{j} n^{k-j} \tag{2.3}$$

so that Theorem 2.3 follows from Theorem 2.2. Now

$$(i+1)(q_{i+1}(m) - q_i(m)) \leq \sum_{k=i}^{m} a_k \binom{k}{i} n^{k-i-1} [k - (n+1)i - n]. \tag{2.4}$$

Suppose $m_* \leq i \leq m - 1$. Then

$$k - (n+1)i - n \leq m - (n+1)i - n \leq m - (n+1)m_* - n \leq 0, \tag{2.5}$$

where we have employed the Lemma in the last step. We conclude that every term in the sum (2.4) is nonpositive. Thus for $m_* \leq i \leq m - 1$ we have $q_{i+1}(m) \leq q_i(m)$.

Now assume $0 \leq i \leq m_* - 1$. We show that $q_{i+1}(m) \geq q_i(m)$. Observe that in this case the sum (2.4) contains terms of both signs, so the positivity of the sum is not apriori clear. Consider
(i + 1) (q_{i+1}(m) - q_i(m)) = \sum_{k=(n+1)i+1}^{m} a_k \binom{k}{i} n^{k-i-1} [k - (n+1)i - n] \\
- \sum_{k=i}^{n} a_k \binom{k}{i} n^{k-i-1} [-k + (n+1)i + n] \\
:= T_2 - T_1. \quad (2.6)

Observe that
\[ T_1 = \sum_{k=1}^{(n+1)i+n-1} a_k \binom{k}{i} n^{k-i-1} [-k + (n+1)i + n] \]
\[ \leq a_{(n+1)(i+1)} \sum_{k=i}^{(n+1)i+n-1} \binom{k}{i} n^{(n+1)i+n-1-i-1} [-k + (n+1)i + n] \]
\[ \leq a_{(n+1)(i+1)} n^{(i+1)n-2} \sum_{k=i}^{(n+1)i+n-1} \binom{k}{i} [-k + (n+1)i + n]. \]

The monotonicity of the coefficients of $P$ was used in the first step.

The last sum can be evaluated (e.g. symbolically) as
\[ \sum_{k=i}^{(n+1)i+n-1} \binom{k}{i} [-k + (n+1)i + n] = \frac{(n+1)i+n+1)!}{(i+2)! (ni+n-1)!}, \]
so that
\[ T_1 \leq a_{(n+1)(i+1)} n^{(i+1)n} \times \frac{((n+1)i+n+1)!}{n^2 (i+2)! (ni+n-1)!} \]
\[ \leq a_{(n+1)(i+1)} n^{(i+1)n} \times \frac{((n+1)i+n+1)!}{(ni+2n)(ni+n)! (ni+n-1)!}. \]

Now observe that
\[ \frac{((n+1)i+n+1)!}{(ni+2n)(ni+n)! (ni+n-1)!} \leq \binom{(n+1)(i+1)}{i}. \]

The inequality $T_1 \leq T_2$ now follows since the upper bound for $T_1$ established above is the first term in the sum defining $T_2$. 

**Corollary 2.4.** Let $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m$ be a sequence of real numbers, $n \in \mathbb{N}$, and
\[ P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m. \]
Then $P(x+n)$ has decreasing coefficients for $n \geq m$.

**Example 2.5.** Let $2 < a_1 < \cdots < a_p$ and $r_1, \cdots, r_p$ be two sequences of positive integers. Then the sequence
\[ q_j := \sum_{k=j}^{m} n^{k-j} \binom{a_1 m}{k r_1} \binom{a_2 m}{k r_2} \cdots \binom{a_p m}{k r_p} \binom{k}{j}, \quad 0 \leq j \leq m \]
is unimodal.

3. The converse of the original criterion

The original criterion for unimodality states that if \( P(x) \) has positive nondecreasing coefficients, then \( P(x + 1) \) is unimodal. In this section we discuss the following inverse question:

Given a unimodal sequence \( \{d_j : 0 \leq j \leq m\} \), is there a polynomial \( P(x) = a_0 + a_1 x + \cdots + a_m x^m \) with nonnegative nondecreasing coefficients such that

\[
P(x + 1) = \sum_{j=0}^{m} d_j x^j
\]

We begin by expressing the conditions on \( \{a_j\} \) that guaranteed unimodality of \( P(x + 1) \) in terms of the coefficients \( \{d_j\} \). Recall that

\[
d_j = \sum_{k=j}^{m} a_k \binom{k}{j}
\]

and

\[
a_j = \sum_{k=j}^{m} (-1)^{k-j} d_k \binom{k}{j}.
\]

Lemma 3.1. Let \( 0 \leq j \leq m \). Then

\[
a_j \geq 0 \iff d_j \geq \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j}.
\]

Proof. This follows directly from (3.3).

Lemma 3.2. Let \( 0 \leq j \leq m - 1 \). Then

\[
a_j \leq a_{j+1} \iff d_j \leq \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1}.
\]

Proof. This follows directly from the identity

\[
a_{j+1} - a_j = \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1} - d_j.
\]

We now combine the previous two lemmas to produce a criterion for unimodality.

Theorem 3.3. Let \( Q(x) = d_0 + d_1 x + \cdots + d_m x^m \) and assume the coefficients \( \{d_j\} \) satisfy the inequalities

\[
\sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j} \leq d_j \leq \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1}.
\]

Then \( Q(x) \) is a unimodal polynomial for which \( P(x) := Q(x - 1) \) has positive and nondecreasing coefficients. Furthermore, for any \( n \in \mathbb{N} \), \( Q(x + n) \) is unimodal with mode \( \lfloor \frac{m}{2} \rfloor \).
Proof. The first part follows from the previous two lemmas. For the second part, Theorem 3.3 shows that $Q(x - 1)$ has nonnegative, nondecreasing coefficients, so Theorem 2.2 yields the result.

\[ \square \]

Note. The inequality (3.5) is always consistent. The difference between the upper and lower bound is

\[
\sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k+1}{j+1} - \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j} = \sum_{k=j+1}^{m} (-1)^{k-j+1} d_k \binom{k}{j+1} = a_{j+1},
\]

so the difference is always nonnegative.

Note. It would be interesting to describe the precise range of the map $(a_0, a_1, \ldots, a_m) \mapsto (d_0, d_1, \ldots, d_m)$. This map is linear, so the image of the set $0 \leq a_0 \leq \cdots \leq a_m$ is a polyhedral cone. In this paper we state one simple restriction on this image.

Proposition 3.4. Let $a_j \geq 0$. Then $d_j \geq d_{j+1}$ for $j \geq \lfloor m/2 \rfloor$.

Proof. This follows directly from

\[
d_j - d_{j+1} = \sum_{k=j}^{m} a_k \binom{k}{j} - \sum_{k=j+1}^{m} a_k \binom{k}{j+1} = a_j + \sum_{k=j+1}^{m} a_k \frac{k!}{(j+1)!(k-j)!} (2j+1-k),
\]

since every term in the last sum is nonnegative. \[ \square \]

4. A criterion for log concavity

Any nonnegative differentiable function $f$ that satisfies $f(0) = f(m) = 0$ and $f''(x) \leq 0$ yields the unimodal sequence \{f(j) : 0 \leq j \leq m\}. The next theorem shows that these sequences are always log concave.

Proposition 4.1. Let $P(x) = \sum_{k=0}^{m} c_k x^k$ be a unimodal polynomial with mode $n$. Assume in addition that $c_{j+1} - 2c_j + c_{j-1} \leq 0$. Then $P(x)$ is log concave.

Proof. Let $j < n$, so that $c_j \geq c_{j-1}$. The condition on $c_j$ can be written as $c_j - c_{j-1} \geq c_{j+1} - c_j$, so that

\[ c_j c_j - c_j c_{j-1} \geq c_{j+1} c_{j-1} - c_j c_{j-1}, \]

and thus the log concavity condition holds. The case $j \geq n$ is similar. \[ \square \]
5. The motivating example

The original criterion for unimodality in [2] was developed in our study of the coefficients \( d_l(m) \) of the polynomial

\[
P_m(a) = \frac{1}{\pi} 2^{m+3/2} (a+1)^{m+1/2} \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}
\]

considered in [1]. These coefficients are given explicitly by

\[
d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{l},
\]

and we have conjectured that \( \{d_l(m)\}_{l=0}^m \) forms a log concave sequence. Unfortunately Proposition 4.1 does not settle this question. For example, for \( m = 15 \) the sequence of signs in \( d_{j+1}(15) - 2d_j(15) + d_{j-1}(15) \), for \( 1 \leq j \leq 14 \), is

\[
\text{sign}(15) = \{+1, +1, +1, +1, +1, -1, -1, -1, -1, +1, +1, +1, +1, +1\},
\]

so the condition fails.

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