A SPECIAL RATIONAL FUNCTION WITH VANISHING INTEGRAL

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Abstract. The integral of a rational function proposed as a question in Mathematics Stack Exchange is evaluated. The integrand has a polynomial of degree 4 as denominator. A natural extension to degree 8 is shown to vanish.

1. Introduction

Question 258746 in Mathematics Stack Exchange asks for the evaluation of

\[ I_1(\alpha, \beta) = \int_{-\infty}^{\infty} \frac{dw}{(\alpha^2 - w^2) + \beta^2 w^2}. \]

It is convenient to expand the integrand and introduce the scaling to obtain

\[ I_1(\alpha, \beta) = 2 \int_{0}^{\infty} \frac{dw}{w^4 + (\beta^2 - 2\alpha^2)w^2 + \alpha^4} = \frac{2}{\alpha^3} \int_{0}^{\infty} \frac{dt}{t^4 + 2at^2 + 1}, \]

where \( a = \beta^2/2\alpha^2 - 1 \). The value of \( I_1(\alpha, \beta) \) is now obtained from the next result.

Theorem 1.1. For \( a > -1 \) and \( m \in \mathbb{N} \cup \{0\} \), define

\[ N_{0,4}(a; m) := \int_{0}^{\infty} \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}. \]

Then

\[ N_{0,4}(a; m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a + 1)]^{m+1/2}}, \]

where \( P_m(a) \) is the polynomial

\[ P_m(a) = \sum_{\ell=0}^{m} \left[ 2^{-2m} \sum_{k=\ell}^{m} 2^k \binom{2m - 2k}{m - k} \binom{m + k}{k} \binom{k}{\ell} \right] a^{\ell}. \]

The special case \( m = 0 \) gives

\[ N_{0,4}(a; 0) = \int_{0}^{\infty} \frac{dx}{x^4 + 2ax^2 + 1} = \frac{\pi}{2\sqrt{2}\sqrt{a + 1}} \]

and this produces

\[ I_1(\alpha, \beta) = \frac{\pi}{\alpha^2 \beta}. \]

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directly.

An elementary proof of Theorem 1.1, using only the value of the Wallis’ integral
\[
\int_0^\infty \frac{dx}{(x^2 + 1)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m},
\]
appears in [4]. The reader will find in [1] a variety of proofs, including the original
one in [2].

2. AN EMAIL REQUEST

In a recent email, the author was asked for the evaluation of
\[
I_2(p, q) = \int_{-\infty}^{\infty} \frac{x^4 - p + qx^2}{(x^4 - p)^2 + (qx^2)^2} \, dx, \quad \text{for } p, q > 0.
\]
This is a natural extension of the original question about the integral $I_1$ in (1.1).

A brute force computation of $I_2(p, q)$ using Mathematica gives
\[
I_2(1, 1) = 0 \text{ and } I_2(2, 3) = 0.
\]
On the other hand, if one asks for the value of $I_2(p, q)$ with $p$ and $q$ kept as parameters,
produces a result with a variety of restrictions such as
\[
\text{Re} \left( p - \frac{1}{2}q^2 - \frac{1}{2} \sqrt{-4pq^2 + q^4} \right)^{1/4} > 0.
\]
This is not a natural restriction, since (2.1) converges for any value $p, q > 0$.

Symbolic examples suggest that $I_2(p, q) = 0$. But more seems to be true. Let
\[
f(x; p, q) = \frac{x^4 - p + qx^2}{(x^4 - p)^2 + (qx^2)^2}
\]
then the examples above satisfy
\[
\int_0^1 f(x; p, q) \, dx = -\int_1^\infty f(x; p, q) \, dx,
\]
and this gives $I_2(p, q) = 0$. An elementary approach to (2.5), following techniques
developed in the classical book [5], is presented next.

Lemma 2.1. Assume $g(x)$ satisfies $g(1/x) = -x^2 g(x)$. Then $\int_0^\infty g(x) \, dx = 0$.

Proof. Split the integral into $[0, 1]$ and $[1, \infty)$ and make the change of variables
$x \mapsto 1/x$ in the second interval. \hfill \Box

It is unfortunate that $f(x; p, q)$ does not satisfy the hypothesis of Lemma 2.1. A
different approach is required. This is presented next.

Expanding the denominator in (2.1) and using the symmetry of the integrand

\[
I_2(p, q) = 2 \int_0^\infty \frac{x^4 + qx^2 - p}{x^8 + (q^2 - 2p)x^4 + p^2} \, dx.
\]

In order to compute $I_2(p, q)$ introduce the notation
\[
T_k(a) = \int_0^\infty \frac{t^k \, dt}{t^8 + 2at^4 + 1}.
\]
Lemma 2.2. The integral $I_2(p,q)$ is given by

\[ I_2(p,q) = 2p^{-3/4}T_4(a) + 2qp^{-5/4}T_2(a) - 2p^{-3/4}T_0(a) \]

with $a = q^2/2p - 1$.

Proof. Make the change of variables $x = p^{1/4}t$ so that

\[ x^8 + (q^2 - 2p)x^4 + p^2 = p^2 \left( t^8 + 2at^4 + 1 \right). \]

The rest is elementary. \qed

The integrals $T_k(a)$ are evaluated in the next section.

3. The integrals $T_k$

This section presents the evaluation of the integrals $T_k$. The first result was established in [3]. The conditions $a_1 > \max \{-a_2 - 1, -\text{sign}(a_2 + 4) \times (a_2^2/8 + 1)\}$ guarantee the convergence of the integral below. In particular, if $a_2 = 0$ this becomes $a_1 > -1$.

Theorem 3.1. Define

\[ M_s(a_1, a_2; r) := \int_0^\infty \left( \frac{x^4}{x^8 + a_2x^6 + 2a_1x^4 + a_2x^2 + 1} \right)^r dx, \]

where $r \in \mathbb{N}$. Then

\[ M_s(a_1, a_2; r) = c^{1/4-r}N_{0,4} \left( \frac{a_2 + 4}{2\sqrt{c}}; r - 1 \right), \]

where $c = 2(a_1 + a_2 + 1)$.

Proof. The change of variable $x \mapsto 1/x$ yields a new form of the integral $M_s$:

\[ M_s(a_1, a_2; r) = \int_0^\infty \left( \frac{x^4}{x^8 + a_2x^6 + 2a_1x^4 + a_2x^2 + 1} \right)^r \frac{dx}{x^2}. \]

Computing the average of these two forms and letting $x = \tan \theta$ and then $\psi = 2\theta$ produces

\[ M_s(a_1, a_2; r) = 2^{-r+1}\int_0^\pi (1 - C)^{2r-1} d\psi \]

\[ \left[ (a_1 - a_2 + 1)C^2 + 7(2 - a_1 - a_2)C + (17 + 3a_2 + a_1) \right]^r, \]

where $C = \cos \psi$. The substitution $z = \cot \psi$ then gives

\[ M_s(a_1, a_2; r) = 2^{-r+1}\int_0^\infty \frac{dz}{(8z^4 + 2(a_2 + 4)z^2 + (a_1 + a_2 + 1))^r}. \]

The change of variable $z \mapsto (8/(a_1 + a_2 + 1))^{1/4}t$ and scaling (1.6) yield (3.2). \qed

The special case $a_2 = 0$ and $r = 1$ gives the value of $T_4(a)$.

Corollary 3.2. The integral $T_4(a)$ is given by

\[ T_4(a) = \frac{\pi}{2^{9/4}\sqrt{a + 1} \sqrt{\sqrt{2} + \sqrt{a + 1}}} = \pi \frac{\sqrt{2 - \sqrt{1 + a}}^{1/2}}{2^{9/4}\sqrt{\sqrt{1 - a^2}}}. \]

Proof. Theorem 3.1 gives

\[ T_4(a) = c^{1/4}N_{0,4} \left( \frac{2}{\sqrt{c}}; 0 \right), \]

and the result follows from (1.6). \qed
Corollary 3.3. For \( a > -1 \), the identity \( T_2(a) = T_4(a) \) holds.

Proof. The change of variables \( x \mapsto 1/x \) gives the result. \( \square \)

It does not seem possible to obtain an expression for the remaining integral

\[
T_0(a) = \int_0^\infty \frac{dx}{x^8 + 2ax^4 + 1}
\]

by the previous methods. For a different approach, let \( t = x^4 \) to obtain

\[
T_0(a) = \frac{1}{4} \int_0^\infty \frac{t^{-3/4} dt}{t^2 + 2at + 1}.
\]

This integral is a special case of entry 3.252.11 in [6] where \( P_\mu^\nu(z) \) is the associated Legendre function. This is a special function with hypergeometric representation

\[
P_\mu^\nu(a) = \frac{1}{\Gamma(1-\mu)} \left( \frac{a + 1}{a - 1} \right)^{\mu/2} _2F_1 \left( \frac{-\nu, \nu + 1}{1 - \mu} \mid \frac{1 - a}{2} \right)
\]

given in entry 8.702 of [6]. This yields

\[
T_0(a) = \frac{3\pi}{8\sqrt{a + 1}} _2F_1 \left( \frac{-1, 5}{3} \mid \frac{1 - a}{2} \right).
\]

The functional equation \( \Gamma(x)\Gamma(1-x) = \pi/\sin \pi x \) and \( \Gamma(x + 1) = x\Gamma(x) \) have been used in the simplification.

The final step uses entry 9.121.30 of [6]

\[
_2F_1 \left( \frac{1 + \frac{n}{2}, 1 - \frac{n}{2}}{\frac{3}{2}} \mid z^2 \right) = \frac{\sin(n \arcsin z)}{nz \sqrt{1 - z^2}}
\]

with \( n = 5/2 \) and \( z = \sqrt{(1 - a)/2} \) to obtain

\[
T_0(a) = \frac{\pi \sqrt{2}}{4\sqrt{1 - a^2}} \sin \left( \frac{3}{2} \arcsin \sqrt{\frac{1 - a}{2}} \right).
\]

Using the identity \( \sin(3u) = 3\sin u - 4\sin^3 u \) gives the final expression for \( T_0(a) \).

Proposition 3.4. The integral \( T_0(a) \) is given by

\[
T_0(a) = \frac{\pi}{2^{9/4}\sqrt{1 - a^2}} \left[ \sqrt{2} - \sqrt{1 + a} \right]^{1/2} \left[ 1 + \sqrt{2}\sqrt{1 + a} \right] = T_4(a) \left[ 1 + \sqrt{2}\sqrt{1 + a} \right].
\]
The values of the integrals $T_k(a)$ produce the value of $I_2(p, q)$.

**Theorem 3.5.** Let $p, q > 0$. Then the integral

\begin{equation}
I_2(p, q) = \int_{-\infty}^{\infty} \frac{x^4 - p + qx^2}{(x^4 - p)^2 + (qx^2)^2} \, dx
\end{equation}

vanishes.

**Proof.** Lemma 2.2 is now used to evaluate $I_2(p, q)$ with $a = q^2/2p^2 - 1$. The factor

\begin{equation}
1 + \sqrt{2\sqrt{1 + a}} = 1 + q/\sqrt{p}
\end{equation}

gives

\begin{equation}
I_2(p, q) = 2p^{-3/4}T_4(a) + 2qp^{-5/4}T_2(a) - 2p^{-3/4}T_0(a) = 2T_4(a) \left[ p^{-3/4} + qp^{-5/4} - p^{-3/4} (1 + q/\sqrt{p}) \right] = 0,
\end{equation}

as claimed. \qed

The values of $T_k(a)$ gives a generalization of the vanishing of $I_2(p, q)$.

**Theorem 3.6.** Assume $(A\sqrt{p} + B)p + (\sqrt{p} + q)C = 0$. Then

\begin{equation}
\int_{-\infty}^{\infty} \frac{Ax^4 + Bx^2 + C}{(x^4 - p)^2 + (qx^2)^2} \, dx = 0.
\end{equation}

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**References**


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