THE INTEGRALS IN GRADSHTEYN AND RYZHIK.  
PART 2: ELEMENTARY LOGARITHMIC INTEGRALS

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Abstract. We describe methods to evaluate elementary logarithmic integrals. The integrand is the product of a rational function and a linear polynomial in \( \ln x \).

1. Introduction

The table of integrals by I. M. Gradshteyn and I. M. Ryzhik [3] contains a large selection of definite integrals of the form

\[
\int_a^b R(x) \ln^m x \, dx,
\]

where \( R(x) \) is a rational function, \( a, b \in \mathbb{R}^+ \) and \( m \in \mathbb{N} \). We call integrals of the form (1.1) elementary logarithmic integrals. The goal of this note is to present methods to evaluate them. We may assume that \( a = 0 \) using

\[
\int_a^b R(x) \ln^m x \, dx = \int_0^b R(x) \ln^m x \, dx - \int_0^a R(x) \ln^m x \, dx.
\]

Section 2 describes the situation when \( R \) is a polynomial. Section 3 presents the case in which the rational function has a single simple pole. Finally section 4 considers the case of multiple poles.

2. Polynomials examples

The first example considered here is

\[
I(P; b, m) := \int_0^b P(x) \ln^m x \, dx,
\]

where \( P \) is a polynomial. This can be evaluated in elementary terms. Indeed, \( I(P; b, m) \) is a linear combination of

\[
\int_0^b x^j \ln^m x \, dx,
\]

and the change of variables \( x = bt \) yields

\[
\int_0^b x^j \ln^m x \, dx = b^{j+1} \sum_{k=0}^m \binom{m}{k} \ln^{m-k} b \int_0^1 t^j \ln^k t \, dt.
\]

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The last integral evaluates to \((-1)^k k!/(j+1)^{k+1}\) either an easy induction argument or by the change of variables \(t = e^{-s}\) that gives it as a value of the gamma function.

**Theorem 2.1.** Let \(P(x)\) be a polynomial given by

\[
P(x) = \sum_{j=0}^{p} a_j x^j.
\]

Then

\[
I(P; b, m) := \int_0^b P(x) \ln^m x \, dx = \sum_{k=0}^{m} (-1)^k k! \binom{m}{k} \ln^{m-k} b \sum_{j=0}^{p} a_j \frac{b^{j+1}}{(j+1)^{k+1}}.
\]

This expression shows that \(I(P; b, m)\) is a linear combination of \(b^j \ln^k b\), with \(1 \leq j \leq 1 + p(= 1 + \deg(P))\) and \(0 \leq k \leq m\).

### 3. Linear denominators

We now consider the integral

\[
f(b; r) := \int_0^b \frac{\ln x \, dx}{x + r}
\]

for \(b, r > 0\). This corresponds to the case in which the rational function in (1.1) has a single simple pole.

The change of variables \(x = rt\) produces

\[
\int_0^b \frac{\ln x \, dx}{x + r} = \ln r \ln(1 + b/r) + \int_0^{b/r} \frac{\ln t \, dt}{1 + t}.
\]

Therefore, it suffices to consider the function

\[
g(b) := \int_0^b \frac{\ln t \, dt}{1 + t},
\]

as we have

\[
f(b; r) = \ln r \ln \left(1 + \frac{b}{r}\right) + g\left(\frac{b}{r}\right).
\]

Before we present a discussion of the function \(g\), we describe some elementary consequences of (3.2).

**Elementary examples.** The special case \(r = b\) in (3.2) yields

\[
\int_0^b \frac{dx}{x + b} = \ln 2 \ln b + \int_0^1 \frac{\ln t \, dt}{1 + t}.
\]

Expanding \(1/(1 + t)\) as a geometric series, we obtain

\[
\int_0^1 \frac{\ln t \, dt}{1 + t} = -\frac{1}{2} \zeta(2) = -\frac{\pi^2}{12}.
\]

This appears as \(4.231.1\) in [3]. Differentiating (3.2) with respect to \(r\) produces

\[
\int_0^b \frac{\ln x \, dx}{(x + r)^2} = -\frac{\ln(b + r)}{r} + \frac{\ln r}{r} + \frac{b \ln b}{r(r + b)}.
\]
As \( b, r \to 1 \) we obtain

\[
\int_0^1 \frac{\ln x \, dx}{(1 + x)^2} = -\ln 2.
\]

This appears as \( 4.231.6 \) in [3]. On the other hand, as \( b \to \infty \) we recover \( 4.231.5 \) in [3]:

\[
\int_0^\infty \frac{\ln x \, dx}{(x + r)^2} = \frac{\ln r}{r}.
\]

**The polylogarithm function.** The evaluation of the integral

\[
g(b) := \int_0^b \frac{\ln t \, dt}{1 + t},
\]

requires the transcendental function

\[
\text{Li}_n(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^n}.
\]

This is the *polylogarithm function* and it has also appeared in [5] in our discussion of the family

\[
h_n(a) := \int_0^\infty \frac{\ln^n x \, dx}{(x - 1)(x + a)}, \quad n \in \mathbb{R}, \ a > 0.
\]

In the current context we have \( n = 2 \) and we are dealing with the *dilogarithm function* \( \text{Li}_2(x) \).

**Lemma 3.1.** The function \( g(b) \) is given by

\[
g(b) = \ln b \ln(1 + b) + \text{Li}_2(-b).
\]

**Proof.** The change of variables \( t = bs \) yields

\[
g(b) = \ln b \ln(1 + b) + \int_0^1 \ln s \, ds
\]

Expanding the integrand in a geometric series yields the final identity. \( \square \)

**Theorem 3.2.** Let \( b, r > 0 \). Then

\[
\int_0^b \frac{\ln x \, dx}{x + r} = \ln b \ln \left( \frac{b + r}{r} \right) + \text{Li}_2 \left( -\frac{b}{r} \right).
\]

**Corollary 3.3.** Let \( b > 0 \). Then

\[
\int_0^b \frac{\ln x \, dx}{x + a} = \ln 2 \ln b - \frac{\pi^2}{12}.
\]

**Proof.** Let \( r \to b \) in Theorem 3.2 and use

\[
\text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.
\]

The expression in Theorem 3.2 and the method of partial fractions gives the explicit evaluation of elementary logarithmic integrals where the rational function has simple poles. For example:
Corollary 3.4. Let $0 < a < b$ and $r_1 \neq r_2 \in \mathbb{R}^+$. Then, with $r = r_2 - r_1$, we have
\[
\int_a^b \frac{\ln x \, dx}{(x + r_1)(x + r_2)} = \frac{1}{r} \left[ \ln b \left( \frac{r_2(b + r_1)}{r_1(b + r_2)} \right) + \ln a \left( \frac{r_1(a + r_2)}{r_2(a + r_1)} \right) \right] + \frac{1}{r} \left[ \text{Li}_2 \left( -\frac{b}{r_1} \right) - \text{Li}_2 \left( -\frac{a}{r_1} \right) - \text{Li}_2 \left( -\frac{b}{r_2} \right) + \text{Li}_2 \left( -\frac{a}{r_2} \right) \right].
\]

The special case $a = r_1$ and $b = r_2$ is of interest:

Corollary 3.5. Let $0 < a < b$. Then
\[
\int_a^b \frac{\ln x \, dx}{(x + a)(x + b)} = \frac{1}{b - a} \left[ \ln(ab) \ln(a + b) - \ln 2 \ln(ab) - 2 \ln a \ln b \right] + \frac{1}{b - a} \left[ -2 \text{Li}_2(-1) + \text{Li}_2 \left( -\frac{b}{a} \right) + \text{Li}_2 \left( -\frac{a}{b} \right) \right].
\]

The integral in Corollary 3.5 appears as 4.232.1 in [3]. An interesting problem is to derive 4.232.2 directly from Corollary 3.5.

(3.18) \[ \int_0^\infty \frac{\ln x \, dx}{(x + u)(x + v)} = \frac{\ln^2 u - \ln^2 v}{2(u - v)} \]

directly from Corollary 3.5.

We now present an elementary evaluation of this integral and obtain from it an identity of Euler. We prove that
\[
(3.19) \int_a^b \frac{\ln x \, dx}{(x + a)(x + b)} = \frac{\ln ab}{2(b - a)} \ln \frac{(a + b)^2}{4ab}.
\]

Proof. The partial fraction decomposition
\[
\frac{1}{(x + a)(x + b)} = \frac{1}{b - a} \left( \frac{1}{x + a} - \frac{1}{x + b} \right),
\]
reduces the problem to the evaluation of
\[
I_1 = \int_a^b \frac{\ln x \, dx}{x + a} \quad \text{and} \quad I_2 = \int_a^b \frac{\ln x \, dx}{x + b}.
\]

The change of variables $x = at$ gives, with $c = b/a$,
\[
I_1 = \int_1^c \frac{\ln(at) \, dt}{1 + t} = \ln a \int_1^c \frac{dt}{1 + t} + \int_1^c \frac{\ln t \, dt}{1 + t} = \ln a \ln(1 + c) - \ln a \ln 2 + \int_1^c \frac{\ln t \, dt}{1 + t}.
\]

Similarly,
\[
I_2 = \ln b \ln 2 - \ln b \ln(1 + 1/c) + \int_1^{1/c} \frac{\ln t \, dt}{1 + t}.
\]
Therefore
\[ I_1 - I_2 = \ln a \ln(1 + c) + \ln b \ln(1 + 1/c) - \ln 2 \ln a - \ln 2 \ln b + \int_1^c \frac{\ln t}{1 + t} dt - \int_{1/c}^1 \frac{\ln t}{1 + t} dt. \]

Let \( s = 1/t \) in the second integral to get
\[ \int_{1/c}^1 \frac{\ln t}{1 + t} dt = \int_c^1 \frac{\ln s}{s(1 + s)} ds. \]

Replacing in the expression for \( I_1 - I_2 \) yields
\[ I_1 - I_2 = \ln a (\ln(a + b) - \ln a - \ln 2) - \ln b (\ln 2 - \ln(a + b) + \ln b) + \int_1^c \frac{\ln t}{t} dt. \]

The last integral can now be evaluated by elementary means to produce the result. \( \square \)

Now comparing the two evaluations of the integral in Corollary 3.5 produces an identity for the dilogarithm function.

**Corollary 3.6.** The dilogarithm function satisfies
\[ \text{Li}_2(-z) + \text{Li}_2 \left(-\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2} \ln^2(z). \]

This is the first of many interesting functional equations satisfied by the polylogarithm functions. It was established by L. Euler in 1768. The reader will find in [4] a nice description of them.

4. A SINGLE MULTIPLE POLE

In this section we consider the evaluation of
\[ f_n(b, r) := \int_0^b \frac{\ln x}{(x + r)^n} dx. \]

This corresponds to the elementary rational integrals with a single pole (at \( x = -r \)). The change of variables \( x = rt \) yields
\[ f_n(b, r) = \frac{\ln r}{(n - 1)r^{n-1}} \left[ \frac{(b + r)^{n-1} - r^{n-1}}{(b + r)^{n-1}} \right] + \frac{1}{r^{n-1}} h_n(b/r), \]
where
\[ h_n(b) := \int_0^b \frac{\ln t}{(1 + t)^n} dt. \]

We first establish a recurrence for \( h_n \).

**Theorem 4.1.** Let \( n > 2 \) and \( b > 0 \). Then \( h_n \) satisfies the recurrence
\[ h_n(b) = \frac{n - 2}{n - 1} h_{n-1}(b) + \frac{b \ln b}{(n - 1)(1 + b)^{n-1}} + \frac{1 - (1 + b)^{n-2}}{(n - 1)(n - 2)(1 + b)^{n-2}}. \]
Proof. Start with
\[ h_n(b) = \int_0^b \frac{(1 + t - t) \ln t dt}{(1 + t)^n} = h_{n-1}(b) - \int_0^b \frac{t \ln t dt}{(1 + t)^n}. \]
Integrate by parts in the last integral, with \( u = t \ln t \) and \( dv = dt/(1 + t)^n \) to produce the result. \( \square \)

The initial condition for this recurrence is obtained from the value
\[ h_2(b) = \frac{b}{1 + b} \ln b - \ln(1 + b). \]
This expression follows by a direct integration by parts in
\[ h_2(b) = -\lim_{\epsilon \to 0} \int_{\epsilon}^{b} \frac{d}{dt}(1 + t)^{-1} dt. \]

The first few values of \( h_n(b) \) suggest the introduction of the function
\[ q_n(b) := (1 + b)^{n-1} h_n(b), \]
for \( n \geq 2 \). For example,
\[ q_2(b) = b \ln b - (1 + b) \ln(1 + b). \]

The recurrence for \( h_n \) yields one for \( q_n \).

**Corollary 4.2.** The recurrence
\[ q_n(b) = \frac{(n - 2)}{(n - 1)} (1 + b) q_{n-1}(b) + \frac{b \ln b}{n - 1} - \frac{(1 + b) [(1 + b)^{n-2} - 1]}{(n - 1)(n - 2)}, \]
holds for \( n \geq 2 \).

Corollary 4.2 establishes the existence of functions \( X_n(b), Y_n(b) \) and \( Z_n(b) \), such that
\[ q_n(b) = X_n(b) \ln b + Y_n(b) \ln(1 + b) + Z_n(b). \]
The recurrence (4.8) produces explicit expression for each of these parts.

**Proposition 4.3.** Let \( n \geq 2 \) and \( b > 0 \). Then
\[ X_n(b) = \frac{(1 + b)^{n-1} - 1}{n - 1}. \]
Proof. The function \( X_n \) satisfies the recurrence
\[ X_n(b) = \frac{n - 2}{n - 1} (1 + b) X_{n-1}(b) + \frac{b}{n - 1}. \]
The initial condition is \( X_2(b) = b \). The result is now easily established by induction. \( \square \)

**Proposition 4.4.** Let \( n \geq 2 \) and \( b > 0 \). Then
\[ Y_n(b) = -\frac{(1 + b)^{n-1}}{n - 1}. \]
Proof. The function \( Y_n \) satisfies the recurrence
\[ Y_n(b) = \frac{n - 2}{n - 1} (1 + b) Y_{n-1}(b). \]
This recurrence and the initial condition \( Y_2(b) = -(1 + b) \), yield the result. \( \square \)
It remains to identify the function $Z_n(b)$. It satisfies the recurrence
\begin{equation}
Z_n(b) = \frac{n-2}{n-1}(1+b)Z_{n-1}(b) - \frac{(1+b)[(1+b)^{n-2} - 1]}{(n-2)(n-1)}.
\end{equation}
This recurrence and the initial condition $Z_2(b) = 0$ suggest the definition
\begin{equation}
T_n(b) := -\frac{(n-1)! Z_n(b)}{b(1+b)}.
\end{equation}

**Lemma 4.5.** The function $T_n(b)$ is a polynomial of degree $n - 3$ with positive integer coefficients.

**Proof.** The function $T_n(b)$ satisfies the recurrence
\begin{equation}
T_n(b) = (n-2)(1+b)T_{n-1}(b) + (n-3)! \left[ \frac{(1+b)^{n-2} - 1}{b} \right].
\end{equation}
Now simply observe that the right hand side is a polynomial in $b$. \qed

Properties of the polynomial $T_n(b)$ will be described in future publications. We now simply observe that its coefficients are unimodal. Recall that a polynomial
\begin{equation}
P_n(b) = \sum_{k=0}^{n} c_k b^k
\end{equation}
is called unimodal if there is an index $n^*$, such that $c_k \leq c_{k+1}$ for $0 \leq k \leq n^*$ and $c_k \geq c_{k+1}$ for $n^* < k \leq n$. That is, the sequence of coefficients of $P_n$ has a single peak. Unimodal polynomials appear in many different branches of Mathematics. The reader will find in [2] and [6] information about this property. We now use the result of [1] to establish the unimodality of $T_n$.

**Theorem 4.6.** Suppose $c_k > 0$ is a nondecreasing sequence. Then $P(x+1)$ is unimodal.

Therefore we consider the polynomial $S_n(b) := T_n(b-1)$. It satisfies the recurrence
\begin{equation}
S_n(b) = b(n-2)S_{n-1}(b) + (n-3)! \sum_{r=0}^{n-3} b^r.
\end{equation}
Now write
\begin{equation}
S_n(b) = \sum_{k=0}^{n-3} c_{k,n} b^k,
\end{equation}
and conclude that $c_0,n = (n-3)!$ and
\begin{equation}
c_{k,n} = (n-2)c_{k-1,n-1} + (n-3)!,
\end{equation}
from which it follows that
\begin{equation}
c_{k+1,n} - c_{k,n} = (n-2) [c_{k,n-1} - c_{k-1,n-1}].
\end{equation}
We conclude that $c_{k,n}$ is a nondecreasing sequence.

**Theorem 4.7.** The polynomial $T_n(b)$ is unimodal.
Conclusions. We have given explicit formulas for integrals of the form

\[ \int_{a}^{b} R(x) \ln x \, dx, \]

where \( R \) is a rational function with real poles. Future reports will describe the case of higher powers

\[ \int_{a}^{b} R(x) \ln^m x \, dx, \]

as well as the case of complex poles, based on integrals of the form

\[ C_n(a, r) := \int_{0}^{b} \frac{\ln x \, dx}{(x^2 + r^2)^n}. \]

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References


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