THE INTEGRALS IN GRADSHTEYN AND RYZHIK.
PART 3: COMBINATIONS OF LOGARITHMS AND EXPONENTIALS.

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Abstract. We present the evaluation of a family of exponential-logarithmic integrals. These have integrands of the form $P(e^{tx}, \ln x)$ where $P$ is a polynomial. The examples presented here appear in sections 4.33, 4.34 and 4.35 in the classical table of integrals by I. Gradshteyn and I. Ryzhik.

1. Introduction

This is the third in a series of papers dealing with the evaluation of definite integrals in the table of Gradshteyn and Ryzhik [2]. We consider here problems of the form

\begin{equation}
\int_0^\infty e^{-tx} P(\ln x) \, dx,
\end{equation}

where $t > 0$ is a parameter and $P$ is a polynomial. In future work we deal with the finite interval case

\begin{equation}
\int_a^b e^{-tx} P(\ln x) \, dx,
\end{equation}

where $a, b \in \mathbb{R}^+$ with $a < b$ and $t \in \mathbb{R}$. The classical example

\begin{equation}
\int_0^\infty e^{-x} \ln x \, dx = -\gamma,
\end{equation}

where $\gamma$ is Euler’s constant is part of this family. The integrals of type (1.1) are linear combinations of

\begin{equation}
J_n(t) := \int_0^\infty e^{-tx} (\ln x)^n \, dx.
\end{equation}

The values of these integrals are expressed in terms of the gamma function

\begin{equation}
\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx
\end{equation}

and its derivatives.
2. The evaluation

In this section we consider the value of \( J_n(t) \) defined in (1.4). The change of variables \( s = tx \) yields

\[
J_n(t) = \frac{1}{t} \int_0^\infty e^{-s} (\ln s - \ln t)^n ds.
\]

Expanding the power yields \( J_n \) as a linear combination of

\[
I_m := \int_0^\infty e^{-x} (\ln x)^m dx, \quad 0 \leq m \leq n.
\]

An analytic expression for these integrals can be obtained directly from the representation of the \textit{gamma function} in (1.5).

\textbf{Proposition 2.1.} For \( n \in \mathbb{N} \) we have

\[
\int_0^\infty (\ln x)^n x^{s-1} e^{-x} dx = \left( \frac{d}{ds} \right)^n \Gamma(s).
\]

In particular

\[
I_n := \int_0^\infty (\ln x)^n e^{-x} dx = \Gamma^{(n)}(1).
\]

\textit{Proof.} Differentiate (1.5) \( n \)-times with respect to the parameter \( s \). \( \Box \)

\textbf{Example 2.2.} Formula 4.331.1 in [2] states that\(^1\)

\[
\int_0^\infty e^{-\mu x} \ln x dx = -\frac{\delta}{\mu}
\]

where \( \delta = \gamma + \ln \mu \). This value follows directly by the change of variables \( s = \mu x \) and the classical special value \( \Gamma'(1) = -\gamma \). The reader will find in chapter 9 of [1] details on this constant. In particular, if \( \mu = 1 \), then \( \delta = \gamma \) and we obtain (1.3):

\[
\int_0^\infty e^{-x} \ln x dx = -\gamma.
\]

The change of variables \( x = e^{-t} \) yields the form

\[
\int_{-\infty}^\infty t e^{-t} e^{-e^{-t}} dt = \gamma.
\]

Many of the evaluations are given in terms of the \textit{polygamma function}

\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x).
\]

Properties of \( \psi \) are summarized in Chapter 1 of [4]. A simple representation is

\[
\psi(x) = \lim_{n \to \infty} \left( \ln n - \sum_{k=0}^n \frac{1}{x+k} \right),
\]

from where we conclude that

\[
\psi(1) = \lim_{n \to \infty} \left( \ln n - \sum_{k=1}^n \frac{1}{k} \right) = -\gamma,
\]

\( ^1 \)The table uses \( C \) for the Euler constant.
this being the most common definition of the Euler's constant $\gamma$. This is precisely the identity $\Gamma'(1) = -\gamma$.

The derivatives of $\psi$ satisfy
\begin{equation}
(2.11) \quad \psi^{(m)}(x) = (-1)^{m+1} m! \zeta(m+1, x),
\end{equation}
where
\begin{equation}
(2.12) \quad \zeta(z, q) := \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}
\end{equation}
is the *Hurwitz zeta function*. This function appeared in [3] in the evaluation of some logarithmic integrals.

**Example 2.3.** Formula 4.335.1 in [2] states that
\begin{equation}
(2.13) \quad \int_0^\infty e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left[ \frac{\pi^2}{6} + \delta^2 \right],
\end{equation}
where $\delta = \gamma + \ln \mu$ as before. This can be verified using the procedure described above: the change of variable $s = \mu x$ yields
\begin{equation}
(2.14) \quad \int_0^\infty e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left( I_2 - 2 I_1 \ln \mu + I_0 \ln^2 \mu \right),
\end{equation}
where $I_n$ is defined in (2.4). To complete the evaluation we need some special values: $\Gamma(1) = 1$ is elementary, $\Gamma'(1) = \psi(1) = -\gamma$ appeared above and using (2.11) we have
\begin{equation}
(2.15) \quad \psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \left( \frac{\Gamma'(x)}{\Gamma(x)} \right)^2.
\end{equation}
The value
\begin{equation}
(2.16) \quad \psi'(1) = \zeta(2) = \frac{\pi^2}{6},
\end{equation}
where $\zeta(z) = \zeta(z, 1)$ is the Riemann zeta function, comes directly from (2.11). Thus
\begin{equation}
(2.17) \quad \Gamma''(1) = \zeta(2) + \gamma^2.
\end{equation}

Let $\mu = 1$ in (2.13) to produce
\begin{equation}
(2.18) \quad \int_0^\infty e^{-x} (\ln x)^2 \, dx = \zeta(2) + \gamma^2.
\end{equation}

Similar arguments yields formula 4.335.3 in [2]:
\begin{equation}
(2.19) \quad \int_0^\infty e^{-\mu x} (\ln x)^3 \, dx = -\frac{1}{\mu} \left[ \delta^3 + \frac{1}{2} \pi^2 \delta - \psi''(1) \right],
\end{equation}
where, as usual, $\delta = \gamma + \ln \mu$. The special case $\mu = 1$ now yields
\begin{equation}
(2.20) \quad \int_0^\infty e^{-x} (\ln x)^3 \, dx = -\gamma^3 - \frac{1}{2} \pi^2 \gamma + \psi''(1).
\end{equation}
Using the evaluation
\begin{equation}
(2.21) \quad \psi''(1) = -2\zeta(3)
\end{equation}
produces
\begin{equation}
(2.22) \quad \int_0^\infty e^{-x} (\ln x)^3 \, dx = -\gamma^3 - \frac{1}{2} \pi^2 \gamma - 2\zeta(3).
\end{equation}
Problem 2.4. In [1], page 203, we introduced the notion of weight for some real numbers. In particular, we have assigned $\zeta(j)$ the weight $j$. Differentiation increases the weight by 1, so that $\zeta'(3)$ has weight 4. The task is to check that the integral
\[
I_n := \int_0^\infty e^{-x} (\ln x)^n \, dx
\]
is a homogeneous form of weight $n$.

3. A small variation

Similar arguments are now employed to produce a larger family of integrals. The representation
\[
\int_0^\infty x^{s-1} e^{-\mu x} \, dx = \mu^{-s} \Gamma(s),
\]
is differentiated $n$ times with respect to the parameter $s$ to produce
\[
\int_0^\infty (\ln x)^n x^{s-1} e^{-\mu x} \, dx = \left( \frac{d}{ds} \right)^n [\mu^{-s} \Gamma(s)].
\]
The special case $n = 1$ yields
\[
\int_0^\infty x^{s-1} e^{-\mu x} \ln x \, dx = \frac{d}{ds} [\mu^{-s} \Gamma(s)] = \mu^{-s} (\Gamma'(s) - \ln \Gamma(s)) = \mu^{-s} \Gamma(s) (\psi(s) - \ln \mu).
\]
This evaluation appears as 4.352.1 in [2]. The special case $\mu = 1$ yields
\[
\int_0^\infty x^{s-1} e^{-x} \ln x \, dx = \Gamma'(s),
\]
that is 4.352.4 in [2].

Special values of the gamma function and its derivatives yield more concrete evaluations. For example, the functional equation
\[
\psi(x + 1) = \psi(x) + \frac{1}{x},
\]
that is a direct consequence of $\Gamma(x + 1) = x \Gamma(x)$, yields
\[
\psi(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}.
\]
Replacing $s = n + 1$ in (3.3) we obtain
\[
\int_0^\infty x^n e^{-\mu x} \ln x \, dx = \frac{n!}{\mu^{n+1}} \left( \sum_{k=1}^{n} \frac{1}{k} - \gamma - \ln \mu \right),
\]
that is 4.352.2 in [2].

The final formula of Section 4.352 in [2] is 4.352.3
\[
\int_0^\infty x^{n-1/2} e^{-\mu x} \ln x \, dx = \frac{\sqrt{\pi} (2n - 1)!!}{2^n \mu^{n+1/2}} \left[ 2 \sum_{k=1}^{n} \frac{1}{2k} - \gamma - \ln(4\mu) \right].
\]
This can also be obtained from (3.3) by using the classical values

\[
\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^n} (2n - 1)!!
\]

\[
\psi(n + \frac{1}{2}) = -\gamma + 2 \left( \sum_{k=1}^{n} \frac{1}{2k - 1} - \ln 2 \right).
\]

The details are left to the reader.

Section 4.353 of [2] contains three peculiar combinations of integrands. The first two of them can be verified by the methods described above: formula \textbf{4.353.1} states

\begin{equation}
\int_{0}^{\infty} (x - \nu)x^{\nu-1}e^{-x} \ln x \, dx = \Gamma(\nu),
\end{equation}

and \textbf{4.353.2} is

\begin{equation}
\int_{0}^{\infty} (\mu x - n - \frac{1}{2})x^{n-\frac{1}{2}}e^{-\mu x} \ln x \, dx = \frac{(2n - 1)!!}{(2\mu)^n} \sqrt{\frac{\pi}{\mu}}.
\end{equation}

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\textbf{References}


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