A SIMPLE EXAMPLE OF A NEW CLASS OF LANDEN TRANSFORMATIONS

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1. Introduction

The method of completing squares yields an elementary procedure to evaluate

\[ I = \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}. \]  

Write

\[ ax^2 + bx + c = a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2}, \]

and use a linear change of variables to obtain

\[ \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}. \]

Observe that \(4ac - b^2 > 0\) is required for the convergence of (1.1).

The goal of this paper is to present a new proof of (1.3). We illustrate a technique that will apply to any rational integrand. Providing new proofs of an elementary result, such as (1.3), is usually an effective tool to introduce students to more interesting Mathematics. The method discussed here has a rich history that we describe in Section 2.

It is an unfortunate fact that, despite our best efforts, evaluating definite integrals is not very much in fashion today. Thus we rephrase the previous evaluation as a question in dynamical systems: replace the parameters \(a, b\) and \(c\) in (1.1) with new ones given by the rules

\[ a_{n+1} = a_n \left\{ \frac{(a_n + 3c_n)^2 - 3b_n^2}{(3a_n + c_n)(a_n + 3c_n) - b_n^2} \right\}, \]

\[ b_{n+1} = b_n \left\{ \frac{3(a_n - c_n)^2 - b_n^2}{(3a_n + c_n)(a_n + 3c_n) - b_n^2} \right\}, \]

\[ c_{n+1} = c_n \left\{ \frac{(3a_n + c_n)^2 - 3b_n^2}{(3a_n + c_n)(a_n + 3c_n) - b_n^2} \right\}, \]

with \(a_0 = a, b_0 = b\) and \(c_0 = c\). The reader is asked to check that (1.1) is invariant under (1.4), that is,

\[ \int_{-\infty}^{\infty} \frac{dx}{a_{n+1}x^2 + b_{n+1}x + c_{n+1}} = \int_{-\infty}^{\infty} \frac{dx}{a_nx^2 + b_nx + c_n}. \]
and to prove that
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = \frac{1}{2} \sqrt{4ac - b^2} \quad \text{and} \quad \lim_{n \to \infty} b_n = 0.
\]
Once this is done, we can pass to the limit in (1.5) and use the invariance of \( I \) to obtain
\[
I = \frac{\pi}{\lim_{n \to \infty} a_n}.
\]
This leads directly to a proof of (1.3). The advantage of this method is that it generalizes to integrands of higher degree.

We call (1.4) a \textit{rational Landen transformation} and in Section 2 we discuss the historical precedent and motivation behind them. This connects it to the magic of the arithmetic-geometric mean, \( \pi \), and a wonderful numerical calculation of Gauss.

The rest of the paper is devoted to a detailed proof of the Landen transformation: the invariance of the rational integral (1.5) and the evaluation of the limits in (1.6). A scaling of the integrand that is a crucial step in producing this transformation is presented in Section 3. The following section presents the trigonometrical aspects of this problem and completes the proof of (1.5). An algebraic calculation shows that the discriminant of the quadratic in (1.1) is preserved; that is,
\[
4ac - b^2 = 4a_1c_1 - b_1^2.
\]
This is used in Section 5 to analyze the dynamics of (1.4) and to establish (1.6).

2. \textsc{Landen transformations}

Many of the evaluations encountered in integral calculus illustrate the fact that definite integrals correspond to special values of functions. For example, the last integral in (1.3) is given by \( \pi = \tan^{-1}(\infty) - \tan^{-1}(-\infty) \). Other special values appear in the elementary courses:
\[
\int_0^1 \frac{dx}{\sqrt{3 - x^2}} = \sin^{-1}\left( \frac{1}{\sqrt{3}} \right).
\]
The same is true for more complicated integrals. For instance, for \( 0 < b < a < 1 \),
\[
G(a, b) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \frac{1}{a} K(k),
\]
with \( k^2 = 1 - b^2/a^2 \). Here \( K \) is the \textit{complete elliptic integral of the first kind} defined by
\[
K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.
\]
Elliptic integrals appear at the center of classical analysis. Their name comes from the fact that they provide explicit formulas for the length of an ellipse.

The inverse of
\[
f(z) = \int_0^z \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}
\]
is similar to \( \sin z \), so (2.1) and (2.3) are not so different after all. This new function is the \textit{elliptic sine} or \textit{sinus amplitudinus} of Jacobi [15], denoted by \( \text{sn} \ z \). It completes the trilogy: \( \sin z \) (circular), \( \sinh z \) (hyperbolic) and \( \text{sn} \ z \) (elliptic). The question
of evaluating definite integrals sometimes comes down to how many functions one knows.

Our complaint that students today are exposed only to the most basic of functions is not new. Klein [16] states\footnote{The authors learn this quote from the preface of [4].} When I was a student, abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics and each of us was ambitious to make progress in this field. And now? The younger generation hardly knows abelian functions.

And now for something completely different [20]. It is not hard to check that the iteration

\[ a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_nb_n} \]

converges to a common limit: the arithmetic-geometric mean of \( a \) and \( b \), denoted by \( \text{AGM}(a,b) \). This is a fascinating function; the book [4] explains its connections with modern algorithms for the evaluation of \( \pi \). The reader will find in [1] a survey of maps similar to (2.5) with an extensive bibliography.

At the turn of the 18\textsuperscript{th} century, Gauss [13] was interested in lemniscates and their lengths. After a numerical calculation, he observed that

\[ \frac{1}{\text{AGM}(1, \sqrt{2})} \quad \text{and} \quad \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{1-x^4}} \]

agree to 11 decimal places. (The integral gives the length of a lemniscate.) With remarkable insight, he discovered that the elliptic integral \( G(a,b) \) in (2.2) remains invariant if the parameters \((a,b)\) are replaced by their arithmetic and geometric means; that is,

\[ G(a,b) = G\left(\frac{a+b}{2}, \sqrt{ab}\right). \]

Iterating, passing to the limit and using the invariance of the elliptic integral \( G \) yields

\[ G(a,b) = \frac{\pi}{2\, \text{AGM}(a,b)}. \]

The convergence of the arithmetic-geometric mean iteration (2.5) is quadratic; that is, \( |a_{n+1} - \text{AGM}(a,b)| \leq C|a_n - \text{AGM}(a,b)|^2 \) for some \( C > 0 \). This leads to a rapid evaluation of the elliptic integral \( G(a,b) \).

This iteration has been used for the numerical evaluation of elliptic integrals. See [6, 7, 8, 9, 10, 11] for some details. The algorithm described here could also be used for the numerical evaluation of rational integrals.

It was a pleasant surprise when, in the process of analyzing definite integrals of rational functions, we discovered that

\[ U_6 = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} \, dx \]

admits a similar invariant transformation. We call this a \textit{rational Landen transformation} since (2.5) and (2.8) had been originally discovered by Landen. In the case
of $U_6$ the dynamical system (2.5) is replaced by

\begin{equation}
\begin{align*}
an+1 &= \frac{a_n b_n + 5a_n + 5b_n + 9}{(a_n + b_n + 2)^{4/3}}, \\
b_{n+1} &= \frac{a_n + b_n + 6}{(a_n + b_n + 2)^{2/3}},
\end{align*}
\end{equation}

with similar rules for $c_n$, $d_n$ and $e_n$. The derivation of (2.10) appears in [2].

The sequence $(a_n, b_n)$ converges to $(3, 3)$ precisely for those initial data $(a_0, b_0)$ for which the integral $U_6$ is finite. Moreover, for the numerator parameters, we have $(c_n, d_n, e_n) \to (1, 2, 1)L$, for some $L \in \mathbb{R}$. The convergence of this method is discussed in [2], [12] and [14]. The invariance of $U_6$ yields the identity

\begin{equation}
U_6 = \frac{\pi}{2L}
\end{equation}

exactly as in (2.8). Observe that (1.7) is also of this type: an integral given as the limit of an iterative process. Transformations similar to (2.10) have been produced in [3] for any even rational integrand.

Until now all rational Landen transformations were restricted to even rational functions. In this paper we present the simplest case of a technique that we expect will extend to the general case; see [17] for details.

The identities (1.7) and (2.8) yield **iterative methods** to evaluate the corresponding integrals. For example, the first 4 iterations of the evaluation of

\begin{equation}
I = \int_{-\infty}^{\infty} \frac{dx}{4x^2 + 3x + 1}
\end{equation}

using (1.4) are given below:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$c_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1.0731707317</td>
<td>0.6585365853</td>
<td>1.7317073171</td>
</tr>
<tr>
<td>2</td>
<td>1.3322738087</td>
<td>0.0186646386</td>
<td>1.3136099170</td>
</tr>
<tr>
<td>3</td>
<td>1.3228754233</td>
<td>4.644065 \times 10^{-7}</td>
<td>1.3228758877</td>
</tr>
<tr>
<td>4</td>
<td>1.3228756555</td>
<td>7.154295 \times 10^{-21}</td>
<td>1.3228756555</td>
</tr>
</tbody>
</table>

The example presented in the table exhibits cubic convergence, faster than the convergence of the AGM. The reader will check that the value $a_4$ gives $\sqrt{7}/2$, the exact value of (2.12), correct to ten digits of accuracy.

At the end of the amazing numerical calculation that led him to establish the invariance for the elliptic integral $G(a, b)$, Gauss commented in his diary that this will surely open up a whole new field of analysis. This statement is certainly true. The reader will find in [4] a detailed discussion of how the arithmetic-geometric mean plays a fundamental role in the modern evaluations of the digits of $\pi$. This technique has also been used in [5] to create new and efficient methods to evaluate elementary functions.

Over the years many other proofs of (2.7) have been discovered; some of them can be found in [18]. The authors are particularly fond of the succinct proof by D. Newman [19]: use $x = b \tan \theta$ and follow with $x \mapsto x + \sqrt{x^2 + ab}$. Change of variables is an art.
3. The quadratic case

The goal of this section is to present the algebraic techniques that produce the transformation (1.4). We scale the integrand by multiplying both the numerator and denominator by an appropriate polynomial. This scaling is one of the main ingredients in the formulation of the Landen transformations. The other one will be discussed in the next section.

We are motivated by the identities

\begin{equation}
U(\tan \theta) = -\frac{\sin 3\theta}{\cos^3 \theta} \quad \text{and} \quad V(\tan \theta) = -\frac{\cos 3\theta}{\cos^3 \theta},
\end{equation}

where

\begin{equation}
U(x) = x^3 - 3x \quad \text{and} \quad V(x) = 3x^2 - 1.
\end{equation}

The task is to find coefficients \(z_0, \ldots, z_4\) and \(e_0, e_1, e_2\) such that

\begin{equation}
(a x^2 + b x + c)(z_0 x^4 + z_1 x^3 + z_2 x^2 + z_3 x + z_4)
\end{equation}

can be written as

\begin{equation}
e_0 U^2(x) + e_1 U(x)V(x) + e_2 V^2(x)
\end{equation}

with the unknown coefficients \(z_i\) and \(e_i\) functions of the original parameters \(a, b\) and \(c\). There is so much freedom, it can’t be hard.

Matching (3.3) with (3.4) yields a system of seven equations for the eight unknowns \(z_0, \ldots, z_4; e_0, e_1, e_2\). We use the first five to solve for the coefficients \(z_i\) in terms of \(a, b, c\) and the unknowns \(e_i\). To start, the constant term gives

\begin{equation}
z_4 = c^{-1} e_2.
\end{equation}

Using this value, the linear coefficient equation is \(c z_3 - 3e_1 + bc^{-1} e_2 = 0\), which yields

\begin{equation}
z_3 = c^{-2}(3ce_1 - be_2).
\end{equation}

The next powers produce

\begin{equation}z_2 = c^{-2}(9c^2 e_0 - 3bce_1 + b^2 e_2 - ace_1 - 6c^2 e_2),\end{equation}

and

\begin{equation}z_1 = c^{-4}(-9bc^2 e_0 + 3b^2 ce_1 - 3ac^2 e_1 - 10c^3 e_1 - b^3 e_2 + 2abc e_2 + 6bc^2 e_2),\end{equation}

and finally

\begin{equation}z_0 = c^{-5}(9bc^2 e_0 - 9ac^3 e_0 - 6c^4 e_0 - 3b^3 ce_1 + 6abc e_1 + 10bc^3 e_1 + b^4 e_2
\end{equation}

\begin{equation}-3ab^2 c e_2 + a^2 c^2 e_2 - 6b^2 c^2 e_2 + 6ac^3 e_2 + 9c^4 e_2).
\end{equation}

This leaves the two equations coming from the vanishing of two highest powers; these are now used to find the parameters \(e_i\). Solve the \(x^5\) equation for \(e_2\) in terms of the parameters \(a, b, c\) and \(e_1, e_0\). Replacing this information in the equation for the leading term produces

\begin{equation}b(b^2 - 3(a - c)^2)e_0 = a(3b^2 - (a + 3c)^2)e_1.
\end{equation}

The system has one degree of freedom, which we use to ensure that the \(z_i\) and \(e_i\) are polynomials in the parameters \(a, b, c\). We choose \(e_0 = a((a+3c)^2 - 3b^2)\), from which it follows that \(e_1 = -b(b^2 - 3(a - c)^2)\). In turns this yields \(e_2 = -c(3b^2 - (3a + c)^2)\).
The expressions for the coefficients \( z_i \) reduce to
\[
\begin{align*}
z_0 &= (a + 3c)^2 - 3b^2 \\
z_1 &= 8b(a - 3c) \\
z_2 &= -6a^2 + 10b^2 + 44ac - 6c^2 \\
z_3 &= 8b(c - 3a) \\
z_4 &= (3a + c)^2 - 3b^2 ,
\end{align*}
\]
and to reiterate,
\[
\begin{align*}
e_0 &= a((a + 3c)^2 - 3b^2) \\
e_1 &= b(3(a - c)^2 - b^2) \\
e_2 &= c((3a + c)^2 - 3b^2) .
\end{align*}
\]
In the latter formulas we already see a semblance of the iteration (1.4).

4. Enters trigonometry

In this section we complete the construction of the Landen transformation and establish the invariance of the integral (1.1) under it. We establish the vanishing of a special class of integrals that appear as intermediate steps in this construction.

Start with (1.1) and use the change of variables \( x = \tan \theta \) to produce
\[
(4.1) \quad I = \int_{\pi/2}^{\pi/2} \frac{d\theta}{a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta} .
\]
The identities
\[
\tan^3 \theta - 3 \tan \theta = -\frac{\sin 3\theta}{\cos^3 \theta} \quad \text{and} \quad 3 \tan^2 \theta - 1 = -\frac{\cos 3\theta}{\cos^3 \theta}
\]
that were the reason behind the choices for \( U \) and \( V \) are used to obtain
\[
(4.2) \quad I = \sum_{k=0}^{4} z_{4-k} \int_{-\pi/2}^{\pi/2} \frac{\sin^k \theta \cos^{4-k} \theta \, d\theta}{e_0 \sin^2 3\theta + e_1 \sin 3\theta \cos 3\theta + e_2 \cos^2 3\theta} .
\]
from the integral (1.1) after it has been scaled according to the procedure described in Section 3.

The elementary identities
\[
(4.3) \quad \begin{align*}
\cos^4 \theta &= \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8} \\
\cos^3 \theta \sin \theta &= \frac{1}{4} \sin 4\theta + \frac{1}{4} \sin 2\theta \\
\cos^2 \theta \sin^2 \theta &= \frac{1}{8} - \frac{1}{4} \cos 4\theta \\
\cos \theta \sin^3 \theta &= \frac{1}{4} \sin 2\theta - \frac{1}{8} \sin 4\theta \\
\sin^4 \theta &= \frac{1}{8} \cos 4\theta - \frac{1}{2} \cos 2\theta + \frac{3}{8}
\end{align*}
\]
reduce the previous integrals to a linear combination of
\[
S_k = \int_{-\pi/2}^{\pi/2} \frac{\sin k\theta \, d\theta}{e_0 \sin^2 3\theta + e_1 \sin 3\theta \cos 3\theta + e_2 \cos^2 3\theta}, \quad k = 2 \text{ and } 4 ,
\]
Proof. In $k$ and cos, resulting in the transformation rule (1.4).

The numerator is the imaginary part of $S$, where we have used periodicity to extend the integral to $[0, \pi/2]$. This reduces (4.2) to an integral of the type (4.1) with new coefficients, resulting in the transformation rule (1.4).

**Lemma 4.1.** The integrals $S_k$ and $C_k$ vanish if $k$ is not a multiple of 3.

**Proof.** In $S_k$ replace $\theta$ by $u = \theta + \pi$. Using $\sin(k[u - \pi]) = (-1)^k \sin ku = \sin ku$ and $\cos(k[u - \pi]) = -\cos ku$ we obtain

$$S_k = \int_{\pi/2}^{3\pi/2} \frac{\sin ku du}{e_0 \sin^2 3u + e_1 \sin 3u \cos 3u + e_2 \cos^2 3u}.$$ 

Adding this to the original $S_k$ and using the periodicity of the integrand we get

$$S_k = \frac{1}{2} \int_0^{2\pi} \frac{\sin ku du}{e_0 \sin^2 3u + e_1 \sin 3u \cos 3u + e_2 \cos^2 3u}.$$ 

Now observe that $\sin 3u$ and $\cos 3u$ are invariant under shifts by $2\pi/3$ and $4\pi/3$, so that

$$6S_k = \int_0^{2\pi} \sin ku + \sin(ku - 2\pi k/3) + \sin(ku - 4\pi k/3) du.$$ 

The numerator is the imaginary part of

$$e^{iku} + e^{i(ku-2\pi k/3)} + e^{i(ku-4\pi k/3)} = e^{iku} (1 + e^{-2\pi ki/3} + e^{-4\pi ki/3}),$$

and the last sum is 0 or 3 according to whether 3 divides $k$ or not. \qed

We conclude that the only terms that contribute to (4.2) are the constants in (4.3). Therefore

$$I = \frac{1}{16} \int_0^{2\pi} \frac{3z_4 + z_2 + 3z_0}{e_0 \sin^2 3u + e_1 \sin 3u \cos 3u + e_2 \cos^2 3u} du,$$

where we have used periodicity to extend the integral to $[0, 2\pi]$. The change of variables $\theta = 3u$ gives

$$I = \frac{1}{8} \int_{-\pi/2}^{\pi/2} \frac{3z_4 + z_2 + 3z_0}{e_0 \sin^2 \theta + e_1 \sin \theta \cos \theta + e_2 \cos^2 \theta} d\theta,$$

and we have returned to the original form (4.1) but with different coefficients. The result in (1.5) is obtained by using $x = \tan \theta$ and the identities

\begin{align*}
8e_0 &= a \left( \frac{(3a + c)^2 - 3b^2}{(3a + c)(a + 3c) - b^2} \right) \\
8e_1 &= b \left( 3(a - c)^2 - b^2 / (3a + c)(a + 3c) - b^2 \right) \\
8e_2 &= c \left( (a + 3c)^2 - 3b^2 / (3a + c)(a + 3c) - b^2 \right).
\end{align*}
5. The analysis of convergence

In the last two sections we have shown the invariance of (1.1) under the Landen transformation (1.4). We now conclude by establishing the convergence of its iterates as in (1.6). In particular, we will show that the error
\[ e_n := (a_n - \frac{1}{2} \sqrt{4ac - b^2}, b_n, c_n - \frac{1}{2} \sqrt{4ac - b^2}) \]
satisfies \( e_n \to 0 \) as \( n \to \infty \). Moreover, we show cubic convergence:
\[ \|e_{n+1}\| \leq C\|e_n\|^3, \]
for some \( C > 0 \).

The analysis of convergence is simpler in the variables \( x = a + c \), \( y = b \), and \( z = a - c \). The dynamical system (1.4) now becomes
\[
\begin{align*}
x_{n+1} &= x_n \left[ 4x_n^2 - 3z_n^2 - 3y_n^2 \right], \\
z_{n+1} &= z_n \left[ 4x_n^2 - y_n^2 - z_n^2 \right], \\
y_{n+1} &= y_n \left[ 3z_n^2 - 3y_n^2 \right],
\end{align*}
\]
with initial conditions \( x_0 = x, y_0 = y \) and \( z_0 = z \).

We now prove that
\[
\lim_{n \to \infty} x_n = \sqrt{x^2 - y^2 - z^2}, \quad \text{and} \quad \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = 0,
\]
or
\[
\lim_{n \to \infty} \left( x_n - \sqrt{x^2 - y^2 - z^2} \right)^2 + y_n^2 + z_n^2 = 0.
\]
This is equivalent to (1.6), so it will conclude the proof of convergence.

To complete the change of variables we use the preservation of the discriminant (1.8) to obtain
\[
x_n^2 - y_n^2 - z_n^2 = x^2 - y^2 - z^2 = 4ac - b^2
\]
and write \( w = \sqrt{4ac - b^2} \). The first equation of iteration (5.3) becomes
\[
\begin{align*}
x_{n+1} &= x_n \left[ \frac{x_n^2 + 3w^2}{3x_n^2 + w^2} \right],
\end{align*}
\]
with initial conditions \( x_0 = a + c > 0 \). The required limit in (5.5) is now
\[
\lim_{n \to \infty} x_n(x_n - w) = 0,
\]
with \( y_n \) and \( z_n \) disappearing into the \( w \) constant. The number of variables has been reduced from 3 to 1.

Using one more change of variables, \( q_n = -ix_n/w \), we reduce (5.6) into
\[
\begin{align*}
q_{n+1} &= \frac{q_n^3 - 3q_n}{3q_n^2 - 1} = \frac{U(q_n)}{V(q_n)},
\end{align*}
\]
and we need to prove that \( q_n \to -i \) to satisfy (5.7). (The polynomials \( U \) and \( V \) introduced in Section 3 have miraculously reappeared!) The trigonometric identity
\[
\frac{U(\cot \theta)}{V(\cot \theta)} = \cot(3\theta)
\]
coupled with a representation of the initial condition as

\[(5.10) \quad q_0 = \cot(it)\]

for some \( t \in \mathbb{R}^+ \) shows that (5.8) reduces to

\[(5.11) \quad q_1 = \frac{U(\cot it)}{V(\cot it)} = \cot(3it),\]

and in general,

\[q_n = \cot(3^n it) = -i e^{2\pi 3^n} + 1 \quad e^{2\pi 3^n} - 1.\]

We conclude that \( q_n \to -i \), and therefore \( x_n \to w \), as desired.

To satisfy (5.10), write \( q_0 = -id \) with \( d = (a + c)/\sqrt{4ac - b^2} \). Now recall that \( 4ac - b^2 > 0 \) and an elementary argument shows that \( d \geq 1 \), so we can take

\[(5.12) \quad t = \coth^{-1}(d) = \frac{1}{2} \ln \frac{d + 1}{d - 1}.\]

The fact that the convergence is cubic follows directly from

\[(5.13) \quad |q_n + i| = \frac{2}{e^{2\pi 3^n} - 1},\]

which decreases to 0 like \( e^{-2\pi 3^n} \). This implies (5.2) and completes the proof of convergence.

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References


