

THE p -ADIC VALUATION OF A SEQUENCE OF COEFFICIENTS

The question deals with

$$P_m(a) = \sum_{l=0}^m d_l(m) a^l,$$

with

$$d_l(m) = 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}.$$

The polynomials $P_m(a)$ arise in the formula

$$\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a).$$

We are interested in the divisibility properties of the coefficients $d_l(m)$. The function $S_p(m)$, for p prime, is the sum of the digits of m when written in base p . The p -adic valuation of an integer n is defined as the exact exponent of p that divides n .

We have established the following result:

Theorem. The 2-adic valuation of the constant term $d_0(m)$ is given by

$$\nu_2(d_0(m)) = S_2(m) - 2m.$$

The 2-adic valuation of the linear term $d_1(m)$ is given by

$$\nu_2(d_1(m)) = 1 - 2m + \nu_2 \left(\binom{m+1}{2} \right) + S_2(m).$$

Problem 1. Produce similar formulas for the other coefficients $\nu_2(d_j(m))$.

The case of the prime 3 seems more difficult.

Problem 2. Prove the existence of a sequence of positive integers m_j such that $\nu_3(d_1(m_j)) = 0$. Extensive calculations show that

$$q_j := m_{j+1} - m_j \in \{2, 7, 20, 61, 182, \dots\}$$

where the sequence $\{q_j\}$ above is defined by $q_1 = 2$ and $q_{j+1} = 3q_j + (-1)^{j+1}$. Establish similar results for other odd primes.

REFERENCES

- [1] Boros, G. - Moll, V., Shallit, J.: *The 2-adic valuation of the coefficients of a polynomial*. Revista Scientia, to appear.