Calculus 121: Practice problems for Test 3

The topics that will be included in Test 3 are:
Optimization problems, Evaluation of limits by L'Hopital's rule, Problems with integration.

1) A forest ranger is in a forest 2 miles from a straight road. A car is located 5 miles down the road. If the forest ranger can walk 3 miles per hour in the forest and 4 miles per hour along the road, toward what point on the road should the ranger walk in order to minimize the time needed to walk to the car?

2) An outdoor track has the shape of a rectangle to which a semicircle is added at each end. Suppose the track has to have total perimeter 440 yards. Find the dimensions for the track that maximize the area of the rectangular portion of the track. Repeat the problem to find the dimensions that minimize the area.

3) A Norman window is a window in the shape of a rectangle with a semicircle attached at the top. Assuming the perimeter of the window is 12 feet, find the dimensions that allow the maximum amount of light to enter.

4) Of all the triangles that pass through the point $(1, 1)$ and have two sides lying on the coordinate axes, one has the smallest area. Find it.

5) A wire of length L is cut into two pieces. One piece is bent to form a square, and the other one is bent to form a circle. Determine the minimum possible value for the sum of the areas of the square and the circle.

6) Find the value of the following limits:

$$\lim_{x\to 0} \frac{\pi - 2\text{Arctan } x}{1/x} \qquad \lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \qquad \lim_{x\to\infty} \frac{e^x}{x^2}$$

$$\lim_{x\to 0} \frac{1-\cos 2x}{1-\cos 3x} \qquad \lim_{x\to 1} \frac{4^x - 3^x - 1}{x-1} \qquad \lim_{x\to\infty} \left(\frac{x-1}{x+1}\right)^x$$

$$\lim_{x\to 0} \frac{\tan x - x}{\sin x - x} \qquad \lim_{x\to\infty} x^{1/2}\sin(1/x) \qquad \lim_{x\to 0^+} \frac{\ln x}{\ln \sin x}.$$

7) Use the identity $\cos^2 x = (1 + \cos 2x)/2$ to find

$$\int_0^{\pi/4} \cos^2 x\, dx.$$

Let $t = \frac{\pi}{2} - x$ and add the two integrals. What do you obtain?

8) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

9) Find the area between the curve $y = x^2$ and the line $y = 2x$.

10) The parabola $y = x^2$ divides the inside of the circle $x^2 + y^2 = 8$ into two parts. Find the areas of both parts.

11) Use integrals to show that the area of a sector of angle θ in a circle of radius r is $\frac{1}{2}r^2\theta$.

12) Evaluate the integral of $\cot x$.

13) If f is a continuous function and

$$\int_0^{10} f(x)\, dx = 20,$$

find

$$\int_0^{2} f(5x)\, dx.$$

14) Find the integral of f from $x = 1$ to $x = 5$ if

$$f(x) = \begin{cases} 3 & \text{for } 1 \leq x < 3 \\ 1/x & \text{for } 3 \leq x < 5 \end{cases}$$

15) Find the volume of revolution obtained by rotating the curve $y = x^2$ from $x = 1$ to $x = 3$ about the x-axis.

16) Obtain the volumen of the solid formed by rotating the area under the curve $y = x^3$ between $x = 1$ and $x = 3$ about the line $x = 100$.