MATH 224-01 **TEST # 1, solutions** **Fall 2009**

$\boxed{\mathbf{1}}$ Is the differential equation

$$\frac{dy}{dx} = -\,x^3 y^5$$

linear? Is it a linear homogeneous equation?

Find a formula, or formulas, for *all* solutions of this equation (i.,e., find the *general solution* of the equation). Find its solution satisfying the initial condition $y(-\,2) = -\,1$. What is the *interval of existence* for this solution?

Solution. Of course the equation is non-linear: its right-hand side contains y^5 rather than just y. Being non-linear, it has absolutely no chance of being linear homogeneous (though I have seen students who wrote: no, such and such an equation is not linear, but it is linear homogeneous).

The equation is separable, so we know how to solve it.

First of all we find constant solutions: those for which the right-hand side is equal to 0; in other words, we find the y's for which $y^5 = 0$. It is only one y: $y = 0$. So we have found one solution:

$$y(x) \equiv 0. \tag{1_1}$$

Now, if $y \neq 0$, we divide both sides of the equation by y, multiply them by dt, and integrate:

$$\frac{dy}{y^5} = -\,x^3\,dx, \quad \int \frac{dy}{y^5} = -\int x^3\,dx, \quad -\frac{1}{4y^4} = -\frac{x^4}{4} + C, \quad \frac{1}{y^4} = x^4 + D,$$

where D, just as C, is an arbitrary constant (a different one: $D = -\,4C$). Solving this equation, we get:

$$y = y(x) = \pm\frac{1}{\sqrt[4]{x^4 + D}}\ : \tag{1_2}$$

either it is with the $+$ sign, or with $-$: both are solutions. Formula (1_2), together with (1_1), gives us all solutions of our equation: its *general solution.*

Now we need to find the particular solution satisfying the initial condition $y(-\,2) = -\,1$. Since $-1 \neq 0$, this solution is definitely not given by formula (1_1), it's given by (1_2) with some D. For a solution given by this formula, let us find $y(-\,2)$:

$$y(-\,2) = \pm\frac{1}{\sqrt[4]{(-\,2)^4 + D}} = -\,1.$$

Since $-\,1 < 0$, the sign has to be $-$ (by definition, the fourth root, and every even-power root, is taken as a nonnegative number). So we have:

$$y(x) = -\frac{1}{\sqrt[4]{x^4 + D}}, \qquad y(-\,2) = -\frac{1}{\sqrt[4]{16 + D}} = -\,1.$$

Now we have to find the constant D:

$$\sqrt[4]{16+D}=1, \qquad 16+D=1, \qquad D=-15,$$

$$y(x)=-\frac{1}{\sqrt[4]{x^4-15}}.$$

This expression makes sense for x such that $x^4-15>0$, i.e. for $x>\sqrt[4]{15}$ and for $x<-\sqrt[4]{15}$: for x in two intervals, $(-\infty,\,-\sqrt[4]{15})$ and $(\sqrt[4]{15},\,\infty)$. Which of these intervals is the interval of existence of our solution?

Clearly it has to be the interval that contains the point -2 at which the initial condition is prescribed: the interval $(-\infty,\,-\sqrt[4]{15})$.

So we can write our solution together with its interval of existence:

$$y(x)=-\frac{1}{\sqrt[4]{x^4-15}}, \qquad x\in(-\infty,\,-\sqrt[4]{15}).$$

$\boxed{\mathbf{2}}$ Consider the autonomous equation $\dfrac{dy}{dt}=(y+1)(\alpha-y^2)$ depending on the parameter α.

(a) For each of the values of the parameter $\alpha=-1$, 0, 1, and 2, draw the picture of the phase line, showing the equilibrium points, indicating their types, and showing the direction of the slope in the intervals between the equilibrium points (I left the space for you to draw the phase lines vertically). For each of these values of α, which of the equilibrium points are asymptotically stable, and which unstable?

(b) For $\alpha=-1$, 0, 1, and 2, what is the limit $\lim_{t\to\infty}y(t)$ of the solution of the differential equation $\dfrac{dy}{dt}=(y+1)(\alpha-y^2)$ with the initial condition $y(0)=1/2$? You don't need to solve the equation (the initial-value problem) to answer this question.

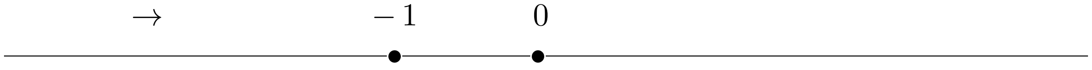
Solution. (a) For $\alpha=-1$ we have only one equilibrium point $y=-1$: $\alpha-y^2=-1-y^2<0$ for all y; $f(y)=(y+1)(-1-y^2)>0$ for $y<-1$, and $f(y)<0$ for y to the right of this point. Let us show this on the phase line, which we'll draw horizontally (cannot have a vertical axis in this file):

$\alpha=-1$

The equilibrium point $y=-1$ is asymptotically stable.

For $\alpha=0$ there are two equilibrium points: $f(y)=(y+1)(-y^2)=0$ for $y=0$ and for $y=-1$. We have three intervals in the y-axis: $(-\infty,\,-1)$, $(-1,\,0)$, and $(0,\,\infty)$. For $y<-1$ we have $y+1<0$, $-y^2<0$, $f(y)=(y+1)(-y^2)>0$, for y between -1 and 0 we have $y+1>0$, $-y^2<0$, $f(y)=(y+1)(-y^2)<0$, and for positive y it is $y+1>0$, $-y^2<0$, and again $f(y)=(y+1)(-y^2)<0$.

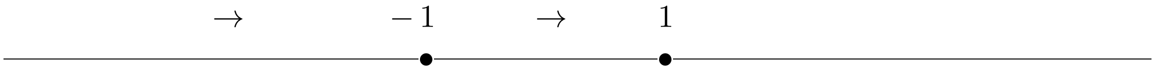
$\alpha=0$



The equilibrium point $y = -1$ is asymptotically stable, and $y = 0$ unstable (if we start a little to the left of it, we go away).

For $\alpha = 1$ there are two equilibrium points: $f(y) = (y+1)(1-y^2) = 0$ for $y = -1$, for $y = 1$. These points divide the y-axis into three intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. For $y < -1$ we have $y + 1 < 0$, $1 - y^2 < 0$, $f(y) = (y+1)(1-y^2) > 0$, we draw a right arrow; for y between -1 and 1 it is $y+1 > 0$, $1-y^2 > 0$, $f(y) = (y+1)(1-y^2) > 0$, again a right arrow; and for y to the right of 1 it is $y + 1 > 0$, $1 - y^2 < 0$, $f(y) = (y - 1.2)(1 - y^2) < 0$, a left arrow:

$\alpha = 1$



The equilibrium point $y = -1$ is unstable, and $y = 1$ asymptotically stable.

Finally, for $\alpha = 2$ the equilibrium points – the roots of the right-hand side $f(y) = (y+1)(\alpha - y^2)$ – are $-\sqrt{2}$, -1, and $\sqrt{2}$. These points divide the real line into four intervals: $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, -1)$, $(-1, \sqrt{2})$, and $(\sqrt{2}, \infty)$. For y in the first of these intervals $y + 1 < 0$, $2 - y^2 < 0$, $f(y) > 0$; in the interval $(-\sqrt{2}, -1)$ we have $y + 1 < 0$, $2 - y^2 > 0$, $f(y) < 0$; for $y \in (-1, \sqrt{2})$ it is $y + 1 > 0$, $2 - y^2 > 0$, $f(y) > 0$; and finally, if $y \in (\sqrt{2}, \infty)$, we have $y + 1 > 0$, $2 - y^2 < 0$, $f(y) < 0$; the picture of the phase line is as follows:

$\alpha = 2$

The equilibrium points $-\sqrt{2}$ and $\sqrt{2}$ are asymptotically stable, the equilibrium point $y = -1$, unstable.

(b) For $\alpha = -1$, from the initial value $y(0) = 1/2$ we start moving down (or to the left), and $\lim_{t\to\infty} y(t) = -1$ (we cannot go past this equilibrium point). For $\alpha = 0$ the initial point is in the interval $(0, \infty)$ (see the picture of the phase line), we start moving to the left, $\lim_{t\to\infty} y(t) = 0$ (note that the limiting point $y = 0$ is an unstable – one-sidedly unstable – equilibrium point in this case; so, this can happen). For $\alpha = 1$, the initial value is in the interval $(-1, 1)$, we start moving to the right, $\lim_{t\to\infty} y(t) = 1$; finally, for $\alpha = 2$ we start moving from the initial point $y(0) = 1/2$ to the right, and the limit as $t \to \infty$ is equal to the equilibrium point $\sqrt{2}$.

Of course, finding (perhaps implicit) formulas for the solution $y(t)$ is possible; but using the phase line is much simpler.

$\boxed{\mathbf{3}}$ Is the differential equation

$$\frac{dy}{dt} = -\frac{y}{t} + 2$$

linear? Is it a linear homogeneous equation? It this equation an autonomous one?

Find the general solution of this equation (i. e., a formula, or formulas, for all solutions of this equation). Find its solution satisfying the initial condition $y(1) = 3$. What is the interval of existence of this solution?

Solution. The equation is linear, it can be represented in the form

$$y' = a(t) \cdot y + b(t)$$

with $a(t) = -1/t$, $b(t) = 2$. It is not linear homogeneous, because $b(t) \not\equiv 0$; it is not autonomous, because $f(t, y) = -\dfrac{y}{t} + 2$ does depend on t.

The integrating factor $u(t) = e^{-\int a(t)\,dt} = e^{\int (1/t)\,dt} = e^{\ln|t|} = |t|$. So we can take as the integrating factor $u(t) = t$ for $t > 0$, and $u(t) = -t$ for $t < 0$. We can also take as the integrating factor the function $u(t) = t$ for all t: it is the same function, only multiplied by some constant for $t < 0$.

Let us multiply our equation $y' = -\dfrac{1}{t} \cdot y + 2$ by the integrating factor:

$$ty' = -y + 2t, \qquad ty' + y = 2t.$$

We see that (as it should be according to our general theory – but we could have made a mistake) the left-hand side is the derivative of the function $t \cdot y(t)$:

$$\frac{d\big(t \cdot y(t)\big)}{dt} = 2t,$$

$$t \cdot y(t) = \int 2t\,dt = t^2 + C;$$

the general solution is

$$y(t) = t + \frac{C}{t},$$

where C is an arbitrary constant.

Now we make use of the initial condition:

$$y(1) = 1 + \frac{C}{1} = 1 + C = 3,$$

$C = 2$, and the particular solution satisfying our initial condition is

$$y(t) = t + \frac{2}{t}.$$

This formula makes sense for all $t \neq 0$, i.e., for $t \in (-\infty, 0) \cup (0, \infty)$: two intervals. The interval of existence is the one containing the point $t = 1$ at which the initial condition is prescribed: the interval $(0, \infty)$.

$\boxed{4}$ A tank contains 100 gallons of water in which 50 pounds of salt have been dissolved. A solution containing 2 pounds of salt per gallon is flowing into the tank at a rate of 10 gallons per minute. The well-mixed solution flows out at the same rate of 10 gallons per minute.

Write a differential equation for $S(t)$, the number of pounds of salt in the tank at time t. With what initial condition should this equation be solved?

Solving the differential equation, find the formula for the amount of salt in the tank after t minutes.

What is the amount of salt in the tank after a very long period of time (mathematical formulation: what is $\lim_{t\to\infty} S(t)$ equal to)?

Solution. The first sentence describes the initial condition:

$$S(0) = 50. \tag{4_1}$$

Clearly the amount of liquid in the tank remains constant (10 gallons in, 10 out every minute); and this constant is equal to 100 (gallons).

Writing the differential equation:

$$\frac{dS}{dt} = 10 \cdot 2 - 10 \cdot \frac{S}{100} = 20 - 0.1S$$

($\dfrac{S(t)}{100}$ represents the concentration of salt in the well-mixed solution: the amount of salt in each gallon of the liquid leaving the tank). This is a linear equation with $a(t) = -0.1$ and $f(t) \equiv 20$ (by the way, an autonomous one, even if it was not asked and won't be used in the solution). As it has been said, the equation has to be solved with the initial condition (4_1).

Solving the equation: the integrating factor $u(t) = e^{-\int a(t)\,dt} = e^{0.1t}$,

$$e^{0.1t} \cdot (S' + 0.1S) = \frac{d}{dt}\big(e^{0.1t} \cdot S(t)\big) = 20\,e^{0.1t},$$

$$e^{0.1t} \cdot S(t) = \int 20\,e^{0.1t}\,dt = 200\,e^{0.1t} + C,$$

$$S(t) = 200 + C\,e^{-0.1t}.$$

This is the general solution.

Taking into account the initial condition to find the particular solution we are interested in:

$$S(0) = 200 + C = 50, \qquad C = -150,$$

$$S(t) = 200 - 150\,e^{-0.1t}.$$

Clearly the limit

$$\lim_{t\to\infty} S(t) = 200.$$

Of course we could find this limit without any differential equations, using only our engineering common sense: since the liquid constantly flows in at the same rate, and out, after a very large time all liquid in the tank will be that which has come in from the tube, having 2 lb. salt per gallon, in 100 gallons 200 pounds.

$\boxed{5}$ Consider a second-order linear homogeneous equation with constant coefficients:

$$y''(t) - 2y'(t) + 5y(t) = 0. \tag{$*$}$$

Of how many functions a fundamental set of solutions must consist?

Write a fundamental set of solutions for this equation.

Write the general solution of the above equation.

Find the particular solution $y(t)$ satisfying the initial conditions $y(0) = -2$, $y'(0) = 0$. Find its value $y(\pi)$ at the point $t = \pi$.

Solution. According to our general theory, a fundamental set of solutions of a *second*-order linear differential equation must consist of *two* solutions.

According to the same theory, we look for solutions in the form $y(t) = e^{\lambda t}$, writing and solving the characteristic equation:

$$\lambda^2 - 2\lambda + 5 = 0, \qquad \lambda_{1,2} = \frac{2 \pm \sqrt{2^2 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

So the fundamental set of solutions is

$$y_1(t) = e^{(1+2i)t}, \qquad y_2(t) = e^{(1-2i)t}.$$

This is true, and can be used to solve what remains of the problem; but it is easier to work with real functions, and take as (another) fundamental set of solutions

$$y_{\text{real}\,1}(t) = e^t \cos 2t, \qquad y_{\text{real}\,2}(t) = e^t \sin 2t$$

(again by our general theory).

The general solution has the form

$$y(t) = C_1 \cdot y_{\text{real}\,1}(t) + C_2 \cdot y_{\text{real}\,2}(t) = e^t(C_1 \cdot \cos 2t + C_2 \cdot \sin 2t),$$

where C_1 and C_2 are arbitrary constants.

We can also write the general solution in the complex form:

$$y(t) = C_{\text{compl.}\,1} \cdot e^{(1+2i)t} + C_{\text{compl.}\,2} \cdot e^{(1-2i)t},$$

where $C_{\text{compl.}\,1}$, $C_{\text{compl.}\,2}$ are *complex* constants; but this is less convenient.

Finding the solution satisfying our initial conditions:

$$y(0) = C_1 \cdot \cos 0 + C_2 \cdot \sin 0 = C_1 = -2;$$

$$y'(t) = e^t(C_1 \cdot \cos 2t + C_2 \cdot \sin 2t - 2C_1 \cdot \sin 2t + 2C_2 \cdot \cos 2t),$$

$$y'(0) = C_1 + 2C_2 = 0, \qquad C_2 = -C_1/2 = 1,$$

$$y(t) = e^t(\sin 2t - 2\cos 2t), \qquad y(\pi) = e^\pi(\sin 2\pi - 2\cos 2\pi) = -2e^\pi.$$