

Lecture note 31. Application of the theory of martingales to time-homogeneous diffusion processes.

It is clear that we need some order.

Microtheorem 31.1. *Let $m(\mathbf{x})$ be a solution of the Dirichlet problem (30.12). Suppose that $m(\mathbf{x})$, defined, originally, only for $\mathbf{x} \in G \cup \partial G$, can be extended to the whole \mathbb{R}^r as a bounded twice continuously differentiable function $u(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^r$ ($u(\mathbf{x}) = m(\mathbf{x})$ for $\mathbf{x} \in G \cup \partial G$), such that the function $Lu(\mathbf{x})$ and the product $(\nabla u(\mathbf{x}))^T \sigma(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^r$, are bounded.*

Then $E(\tau_G^{\mathbf{x}}) < \infty$ for every $\mathbf{x} \in G$, and $m(\mathbf{x}) = E(\tau_G^{\mathbf{x}})$.

Proof. By Itô's formula,

$$Y_t = u(\mathbf{X}_t^{\mathbf{x}}) - \int_0^t Lu(\mathbf{X}_s^{\mathbf{x}}) ds = \int_0^t \sum_{i=1}^r \sum_{k=1}^n \frac{\partial u}{\partial x^i}(\mathbf{X}_s^{\mathbf{x}}) \cdot \sigma_{ik}(\mathbf{X}_s^{\mathbf{x}}) dW_s^k \quad (31.1)$$

is a martingale. The random variable $\tau = \tau_G^{\mathbf{x}}$ is a stopping time (the first time that a continuous stochastic process $X_t^{\mathbf{x}}$ hits the closed set $\mathbb{R}^r \setminus G$); and for every $t_* \in (0, \infty)$ the minimum $\min(\tau, t_*)$ is a *bounded* stopping time (bounded by the constant t_*). So by Microtheorem 29.1,

$$E(Y_{\min(\tau, t_*)}) = E(Y_0); \quad (31.2)$$

or, by (31.1),

$$E\left(u(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}}) - \int_0^{\min(\tau, t_*)} Lu(\mathbf{X}_s^{\mathbf{x}}) ds\right) = E(u(\mathbf{X}_0^{\mathbf{x}})) = u(\mathbf{x}). \quad (31.3)$$

We have $u(\mathbf{x}) = m(\mathbf{x})$ because $\mathbf{x} \in G$; since $\min(\tau, t_*) \leq \tau$, we have $u(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}}) = m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}})$ and $Lu(\mathbf{X}_s^{\mathbf{x}}) = Lm(\mathbf{X}_s^{\mathbf{x}}) = -1$ for $s < \min(\tau, t_*)$; so

$$m(\mathbf{x}) = E(m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}}) + \min(\tau, t_*)), \quad (31.4)$$

$$E(\min(\tau, t_*)) = m(\mathbf{x}) - E(m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}})). \quad (31.5)$$

It is clear that $\min(\tau, t_*)$ for every $\omega \in \Omega$ is a non-decreasing function of t_* (it is *increasing* up to $t_* = \tau_G^{\mathbf{x}}$, and after that it remains constant; if $\tau_G^{\mathbf{x}} = \infty$, this minimum is equal to t_* , and keeps on growing for all t_*). Clearly the limit

$$\lim_{t_* \rightarrow \infty} \min(\tau_G^{\mathbf{x}}, t_*) = \tau_G^{\mathbf{x}}. \quad (31.6)$$

By the monotone-convergence theorem (Theorem 2.1) we have:

$$E(\tau_G^{\mathbf{x}}) = \lim_{t_* \rightarrow \infty} E(\min(\tau_G^{\mathbf{x}}, t_*)) = m(\mathbf{x}) - \lim_{t_* \rightarrow \infty} E(m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}})). \quad (31.7)$$

The function m (being a restriction of a bounded function u) is bounded, so the limit (31.7) is finite, $E(\tau_G^{\mathbf{x}}) < \infty$.

Now, using the fact that $\tau_G^{\mathbf{x}} < \infty$ almost surely, and the fact that $\mathbf{X}_t^{\mathbf{x}}$ is continuous, we have:

$$\lim_{t_* \rightarrow \infty} \mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}} = \mathbf{X}_{\tau}^{\mathbf{x}}; \quad (31.8)$$

and using the fact that the function m is continuous in $G \cup \partial G$, we get:

$$\lim_{t_* \rightarrow \infty} m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}}) = m(\mathbf{X}_{\tau}^{\mathbf{x}}) = 0, \quad (31.9)$$

because $\mathbf{X}_{\tau}^{\mathbf{x}} \in \partial G$.

The random variables $m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}})$ are all bounded by the same constant $\sup\{|m(\mathbf{x})| : \mathbf{x} \in G \cup \partial G\} (\leq \sup\{|u(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^r\} < \infty)$; so by the dominated-convergence theorem (Theorem 2.2) we have:

$$E(\tau_G^{\mathbf{x}}) = m(\mathbf{x}) - \lim_{t_* \rightarrow \infty} E(m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}})) = m(\mathbf{x}) - E\left(\lim_{t_* \rightarrow \infty} m(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}})\right) = m(\mathbf{x}). \quad (31.10)$$

Microtheorem 31.2. *Let G be a region in \mathbb{R}^r such that $E(\tau_G^{\mathbf{x}}) < \infty$. Let $g(\mathbf{x})$, $\mathbf{x} \in G$, and $\varphi(\mathbf{x})$, $\mathbf{x} \in \partial G$, be bounded functions. Suppose $u(\mathbf{x})$ is a solution of the Dirichlet problem*

$$\begin{aligned} Lu(\mathbf{x}) &= \frac{1}{2} \sum_{i,j=1}^n a_{ij}(\mathbf{x}) \cdot \frac{\partial^2 u}{\partial x^i \partial x^j}(\mathbf{x}) + \sum_{i=1}^n b_i(\mathbf{x}) \cdot \frac{\partial u}{\partial x^i}(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in G, \\ u(\mathbf{x}) &= \varphi(\mathbf{x}), \quad \mathbf{x} \in \partial G. \end{aligned} \quad (31.11)$$

Suppose $u(\mathbf{x})$ can be extended as a twice continuously differentiable function, bounded with its derivatives, to the whole \mathbb{R}^r .

Then

$$u(\mathbf{x}) = E\left(\varphi(\mathbf{X}_{\tau}^{\mathbf{x}}) - \int_0^{\tau} g(\mathbf{X}_s^{\mathbf{x}}) ds\right). \quad (31.12)$$

Proof. Again take the martingale

$$Y_t = u(\mathbf{X}_t^{\mathbf{x}}) - \int_0^t Lu(\mathbf{X}_s^{\mathbf{x}}) ds, \quad (31.13)$$

and again take it at the stopping time $\min(\tau, t_*)$:

$$\begin{aligned} E\left(u(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}}) - \int_0^{\min(\tau, t_*)} Lu(\mathbf{X}_s^{\mathbf{x}}) ds\right) \\ = E\left(u(\mathbf{X}_{\min(\tau, t_*)}^{\mathbf{x}}) - \int_0^{\min(\tau, t_*)} g(\mathbf{X}_s^{\mathbf{x}}) ds\right) = E(Y_0) = u(\mathbf{x}). \end{aligned} \quad (31.14)$$

The random variables under the expectation sign are for every t_* dominated by the random variable

$$\sup_{\mathbf{x}} |u(\mathbf{x})| + \sup_{\mathbf{x}} |g(\mathbf{x})| \cdot \tau, \quad (31.15)$$

which has a finite expectation; and by the dominated-convergence theorem, letting $t_* \rightarrow \infty$, we get

$$u(\mathbf{x}) = E\left(u(\mathbf{X}_\tau^x) - \int_0^\tau g(\mathbf{X}_s^x) ds\right) = E\left(\varphi(\mathbf{X}_\tau^x) - \int_0^\tau g(\mathbf{X}_s^x) ds\right). \quad (31.16)$$

Example 31.1. Let W_t^x be a (one-dimensional) Wiener process starting from a point x at time 0; let (a, b) be an interval containing the point x . If we want to find the expectation $E(\tau_{(a,b)}^x)$ (and to find out whether this expectation is finite), the way to do it is to find this expectation (denote it $m(x)$) at once for *all* points $x \in (a, b)$ (no problem finding this expectation for $x = a$ or b : it is equal to 0, because $\tau_{(a,b)}^a = \tau_{(a,b)}^b = 0$). We look for $m(x)$ as the solution of the boundary-value problem

$$\begin{aligned} \frac{1}{2} m''(x) &= -1, & a < x < b, \\ m(a) &= m(b) = 0. \end{aligned} \quad (31.17)$$

The solution is easily found:

$$m(x) = (b-x)(x-a), \quad a \leq x \leq b \quad (31.18)$$

(check that this is a solution).

The formula (31.18) defines a function that is defined on the whole real line; but it is not bounded. So we change the function outside the interval $[a, b]$, taking, say,

$$u(x) = (b-x)(x-a) \cdot h(x-b) \cdot h(a-x), \quad (31.19)$$

where

$$h(x) = \begin{cases} 1, & x \leq 0, \\ 1 - 10x^3 + 15x^4 - 6x^5, & 0 \leq x \leq 1, \\ 0, & x \geq 1. \end{cases} \quad (31.20)$$

The function u is bounded everywhere with its first two derivatives, and it coincides with m in the interval $[a, b]$. So by Microtheorem 31.1 the expectation $E(\tau_{(a,b)}^x)$ is given by (31.18).

So $\tau = \tau_{(a,b)}^x$ is finite almost surely, and we can speak of the distribution of the exit point W_τ^x : the probabilities $P\{W_\tau^x = a\}$, $P\{W_\tau^x = b\}$.

By Microtheorem 31.2, we look for the expectation $u(x) = E(\varphi(W_\tau^x))$ as the solution of the boundary-value problem

$$\begin{aligned} \frac{1}{2} u''(x) &= 0, & x \in (a, b), \\ u(a) &= \varphi(a), & u(b) = \varphi(b). \end{aligned} \quad (31.21)$$

The second derivative is equal to 0: the function is linear. It is found easily:

$$u(x) = \varphi(a) \cdot \frac{b-x}{b-a} + \varphi(b) \cdot \frac{x-a}{b-a}, \quad a \leq x \leq b \quad (31.22)$$

(check it). Again a function is defined by this formula in the whole real line – smooth, but not bounded; again it can be helped the same way. So the expectation $u(x) = E(\varphi(W_\tau^x))$ is given by (31.22). On the other hand,

$$E(\varphi(W_\tau^x)) = \varphi(a) \cdot P\{W_\tau^x = a\} + \varphi(b) \cdot P\{W_\tau^x = b\}. \quad (31.23)$$

Using the fact that $P\{W_\tau^x = a\} + P\{W_\tau^x = b\} = 1$, we get one linear algebraic equation with one unknown, say, $P\{W_\tau^x = b\}$, which has a unique solution. So we get:

$$P\{W_\tau^x = a\} = \frac{b-x}{b-a}, \quad P\{W_\tau^x = b\} = \frac{x-a}{b-a}, \quad a \leq x \leq b. \quad (31.24)$$

This is the general situation: if we can solve the problem (30.9) for a sufficiently wide class of boundary functions φ , we find the distribution of the boundary point \mathbf{X}_τ^x at which the process \mathbf{X}_t^x leaves the region G .

We can also consider the times $\tau_{(a, \infty)}^x, \tau_{(-\infty, b)}^x$ of leaving an infinite interval containing the initial point x . Are these times almost surely finite? (If they are, there is no question about $W_{\tau_{(a, \infty)}^x}^x, W_{\tau_{(-\infty, b)}^x}^x$: the process can leave a semi-infinite interval only through its only end.)

We have for all $\omega \in \Omega$:

$$\tau_{(a, \infty)}^x = \lim_{b \rightarrow \infty} \tau_{(a, b)}^x. \quad (31.25)$$

As for the event $\{\tau_{(a, \infty)}^x < \infty\}$, it is represented as

$$\{\tau_{(a, \infty)}^x < \infty\} = \bigcup_{b > x} \{W_{\tau_{(a, b)}^x}^x = a\}. \quad (31.26)$$

Indeed, if for some $\omega \in \Omega$ the event in the left-hand side occurs, W_t^x leaves the interval (a, b) at the time $\tau_{(a, \infty)}^x$ through its left end (make a picture). The maximum $\max_{0 \leq t \leq \tau_{(a, \infty)}^x} W_t^x$ is finite, and for b greater than this maximum the event $\{W_{\tau_{(a, b)}^x}^x = a\}$ occurs. So $\{\tau_{(a, \infty)}^x < \infty\} \subseteq \bigcup_{b > x} \{W_{\tau_{(a, b)}^x}^x = a\}$.

The opposite inclusion: if the event in the right-hand side of (31.26) occurs, then for some $b \geq x$ the event $\{W_{\tau_{(a, b)}^x}^x = a\}$ occurs; and at some time the Wiener process *is* at the point a : we have left the interval (a, ∞) .

The family of events in the right-hand side of (31.25) is non-decreasing ($\{W_{\tau_{(a, b_1)}^x}^x = a\} \subseteq \{W_{\tau_{(a, b_2)}^x}^x = a\}$ for $b_2 > b_1$), so

$$P\{\tau_{(a, \infty)}^x < \infty\} = \lim_{b \rightarrow \infty} P\{W_{\tau_{(a, b)}^x}^x = a\} = \lim_{b \rightarrow \infty} \frac{b-x}{b-a} = 1. \quad (31.27)$$

So the Wiener process *does* leave almost surely every semi-finite interval, it does almost surely reach every point a to the left of the starting point x , and every point b to the right of it. It follows from this that almost surely

$$\underline{\lim}_{t \rightarrow \infty} W_t^x = -\infty, \quad \overline{\lim}_{t \rightarrow \infty} W_t^x = \infty \quad (31.28)$$

(make a picture of a function defined for $t \in [0, \infty)$ and oscillating wider and wider as $t \rightarrow \infty$).

So almost surely $\tau_{(a, \infty)}^x < \infty$, $\tau_{(-\infty, b)}^x < \infty$. What about the expectations of these random variables?

From (31.25), using the monotone-convergence theorem, we get:

$$E(\tau_{(a, \infty)}^x) = \lim_{b \rightarrow \infty} E(\tau_{(a, b)}^x) = \lim_{b \rightarrow \infty} (b - x)(x - a) = \infty. \quad (31.29)$$

When we started, in the elementary probability theory course, with expectations, we were shown some examples of random variables having no finite expectations; these examples seemed to be pretty artificial. But here we have quite a natural example of a random variable $\tau = \tau_{(a, \infty)}^x$ having infinite expectation.