Lecture 6. Discrete Markov chains, continued.

Up to now, we were concerned with transition probabilities $p_{yy}^{(n)}$ and first return probabilities $f_{yy}^{(k)}$ for one state $y \in X$. What about $p_{xy}^{(n)}$, $f_{yy}^{(k)}$ with $x \ne y$?

Theorem 6.1. *If* $\sum_{n=0}^{\infty} p_{yy}^{(n)} < \infty$ (i. e. the state y is transient), *then also* $\sum_{n=0}^{\infty} p_{xy}^{(n)} < \infty$ *for any other* $x \in X$. *If there exists* $\lim_{n\to\infty} p_{yy}^{(n)} = p_y$, *then* $\lim_{n\to\infty} p_{xy}^{(n)} = P_x\{\tau_y < \infty\} \cdot p_y$.

Proof. We have for $x \ne y$:

$$p_{xy}^{(n)} = \sum_{k=1}^{n} f_{xy}^{(k)} p_{yy}^{(n-k)}, \tag{6.1}$$

$$\sum_{n=0}^{\infty} p_{xy}^{(n)} = \sum_{n=0}^{\infty}\sum_{k=1}^{n} f_{xy}^{(k)} p_{yy}^{(n-k)} = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} f_{xy}^{(k)} p_{yy}^{(n-k)} \le \sum_{k=1}^{\infty} f_{xy}^{(k)} \cdot \sum_{n=0}^{\infty} p_{yy}^{(n-k)} < \infty \tag{6.2}$$

if the state y is transient. As for $\lim_{n\to\infty} p_{xy}^{(n)}$, we take a k_0 so that $\sum_{k=k_0}^{\infty} f_{xy}^{(k)} < \varepsilon/2$, and an n_0 so that $p_y - \varepsilon/2 < p_{yy}^{(n)} < p_y + \varepsilon/2$ for $n \ge n_0$. Then we have from (6.1) for $n \ge n_0 + k_0$:

$$\begin{aligned}\sum_{k=1}^{\infty} f_{xy}^{(k)} \cdot p_y - \varepsilon &\le \sum_{k=1}^{k_0-1} f_{xy}^{(k)} \cdot p_y - \varepsilon/2 < \sum_{k=1}^{k_0-1} f_{xy}^{(k)} p_{yy}^{(n-k)} \\ &\le p_{xy}^{(n)} = \sum_{k=1}^{k_0-1} f_{xy}^{(k)} p_{yy}^{(n-k)} + \sum_{k=k_0}^{n} f_{xy}^{(k)} p_{yy}^{(n-k)} \le \sum_{k=1}^{k_0-1} f_{xy}^{(k)} p_{yy}^{(n-k)} + \sum_{k=k_0}^{n} f_{xy}^{(k)} \\ &< \sum_{k=1}^{k_0-1} f_{xy}^{(k)} \cdot (p_y + \varepsilon/2) + \varepsilon/2 \le \sum_{k=1}^{\infty} f_{xy}^{(k)} \cdot p_y + \varepsilon.\end{aligned} \tag{6.3}$$

So for sufficiently large n the transition probability $p_{xy}^{(n)}$ differs from $P_x\{\tau_y < \infty\} \cdot p_y$ less than by ε, which proves our statement.

To go further we'll introduce some more classification of states of a Markov chain – *groups* of states.

For x, $y \in X$ we'll write $x \to\to y$ (read: the state y can be reached from x) if $p_{xy}^{(N)} > 0$ for some N. It is clear that $x \to\to x$ (take $N = 0$); for $x \ne y$ we have $x \to\to y$ if and only if there exists a chain of points x_1, ..., $x_{N-1} \in X$ ($N \ge 1$) such that the one-step transition probabilities p_{xx_1}, $p_{x_i x_{i+1}}$ for $1 \le i \le N-2$, and $p_{x_{N-1}y}$ are positive.

Theorem 6.2. *If* $y \to\to x$ *and* $P_x\{\tau_y = \infty\} > 0$, *then the state* y *is transient.*

Proof. We have a chain of states $y \to x_1 \to \ldots \to x_{N-1} \to x$ connecting y and x, with positive transition probability at each step. It is possible that some intermediate

$x_i = y$; then we can take a shorter chain of states $y \to x_{i+1} \to \dots \to x_{N-1} \to x$ still connecting y to x. So we can assume, without loss of generality, that all $x_i \neq y$.

We have the inclusion between events:

$$\{\xi_0 = y,\, \tau_y = \infty\} \supseteq \{\xi_0 = y,\, \xi_1 = x_1, \dots, \xi_{N-1} = x_{N-1},\, \xi_N = x,\, \tau_y = \infty\}. \tag{6.4}$$

From this inclusion we get the inequality between the probabilities:

$$\begin{aligned} P_y\{\tau_y = \infty\} &= P_y\{\xi_0 = y,\, \tau_y = \infty\} \\ &\geq P_y\{\xi_0 = y,\, \xi_1 = x_1, \dots, \xi_{N-1} = x_{N-1},\, \xi_N = x,\, \tau_y = \infty\}. \end{aligned} \tag{6.5}$$

It seems obvious that the probability in the right-hand side is equal to

$$P_y\{\xi_0 = y,\, \xi_1 = x_1, \dots, \xi_{N-1} = x_{N-1},\, \xi_N = x\} \cdot P_x\{\tau_y = \infty\} \qquad (> 0)\,. \tag{6.6}$$

In a well-constructed system of a field of mathematics, things that "seem to be obvious" should be proved reasonably simply; so let us give it a try.

The event in the right-hand side of (6.4) can be represented as

$$\bigcap_{m=N}^{\infty} \{\xi_0 = y,\, \xi_1 = x_1, \dots, \xi_{N-1} = x_{N-1},\, \xi_N = x,\, \tau_y > m\}; \tag{6.7}$$

so by the equality $P(\lim_{m\to\infty} A_m) = \lim_{m\to\infty} P(A_m)$ (see (2008.1 − 2.11)) we have:

$$P_y\{\tau_y = \infty\} \geq \lim_{m\to\infty} P_y\{\xi_0 = y,\, \xi_1 = x_1, \dots, \xi_{N-1} = x_{N-1},\, \xi_N = x,\, \tau_y > m\}. \tag{6.8}$$

The probability under the limit sign is equal to

$$\begin{aligned} &\sum_{x_{N+1}, \dots, x_m \neq y} p_{yx_1} \cdot p_{x_1x_2} \cdot \dots \cdot p_{x_{N-1}x} \cdot p_{xx_{N+1}} \cdot p_{x_{N+1}x_{N+2}} \cdot p_{x_{m-1}x_m} \\ &\qquad = p_{yx_1} \cdot p_{x_1x_2} \cdot \dots \cdot p_{x_{N-1}x} \cdot \sum_{y_1, \dots, y_{m-N} \neq y} p_{xy_1} \cdot p_{y_1y_2} \cdot \dots \cdot p_{y_{m-N-1}x_{m-N}} \\ &\qquad = p_{yx_1} \cdot p_{x_1x_2} \cdot \dots \cdot p_{x_{N-1}x} \cdot P_x\{\tau_y > m - N\}. \end{aligned} \tag{6.9}$$

Limit passage as $m \to \infty$ yields that the right-hand side of (6.5) is indeed equal to (6.6).

So the probability $P_y\{\tau_y = \infty\}$ is positive, the state y is transient.

The relation $\to\to$ between the states is reflexive ($x \to\to x$) and transitive (from $x \to\to y$ and $y \to\to z$ it follows that $x \to\to z$); but in may be not symmetric (in general, $x \to\to y \not\Rightarrow y \to\to x$).

Let us introduce another relation between the states, a symmetric one: $x \sim y$ if $x \to\to y$ and $y \to\to x$. This relation os reflexive, symmetric, and transitive; so it is an *equivalence relation*, and we can divide the state space X into disjoint *classes of equivalence* C_i (finitely many or countably many) such that for every x and y belonging to the same class C_i we have $x \sim y$, and for $x \in C_i$, $y \notin C_i$ it is $x \not\sim y$. Some classes may consist of one state only.

We are going to prove that all states belonging to the same class of equivalence are of the same type: either all of them are transient, or all of them recurrent; either all of them are aperiodic, or all periodic with the same period; either the limit $\lim_{n\to\infty} p_{yy}^{(n)}$ exists for all y in the class, or for no state in the class; and if $\lim_{n\to\infty} p_{yy}^{(n)} = 0$ for one state, then the same is true for all other states in the same class.

But for this, to the next lecture.