MATH 224-01 **PREPARATION TEST # 1** **Spring 2010**

1 Is the differential equation

$$\frac{dy}{dt} = t\,y^2$$

linear? Is it a linear homogeneous equation?

Is the function $y(t) \equiv 0$ a solution of this equation? Is $y(t) \equiv 1$ its solution?

Find the general solution of this equation.

Find its solution satisfying the initial condition $y(2) = 1$. On what interval (containing the point $t = 2$, of course) is this solution defined? What happens with $y(t)$ as t approaches the right-hand end of this interval? What happens with the solution as t approaches the left end of the interval in which it is defined?

2 Consider the autonomous equation $\dfrac{dy}{dt} = y^3(\alpha - y)$.

(a) For each of the values of the parameter α, draw the picture of the phase line, showing the equilibrium points and the direction of the motion in the intervals between the equilibrium points. Which of the equilibrium points are asymptotically stable, and which unstable (the answer may be different for different α's)?

(b) For some value of α, draw the picture in the t-y plane showing constant solutions, and the regions in which the solutions go up or down. Show one solution for each of these regions.

3 Is the differential equation

$$\frac{dy}{dt} = -\frac{y}{t} + 2$$

linear? Is it a linear homogeneous equation? It this equation an autonomous one?

Find the general solution of this equation (i. e., a formula, or formulas, for all solutions of this equation). Find its solution satisfying the initial condition $y(1) = 3$. What is the interval of existence of this solution?

There will be a problem in which you'll have to write a differential equation (and initial condition) describing a practical problem: a problem with tanks and salted water, or one about finance, or one about population dynamics, or a problem about heating or cooling; examples are given below.

4a On the moon colony, there are 5 births and 6 deaths per thousand people per year. One thousand people join the colony each month. The colony begins with zero population in the year 2100.

Denoting by $P(t)$ the population of the colony at time t, write a differential equation for $P(t)$ (be sure that you know in what units you measure time).

What is the order of this equation? Is it a linear or a non-linear equation?

With what initial condition(s?) should it be solved?

What will the population of the colony be in the year 3000?

4b A body having the temperature 100 degrees at the time $t = 0$ is placed into a room, where the temperature is 70 degrees. Write the differential equation for the temperature $T(t)$ of the body after time t. With what initial condition should this equation be solved?

The measurement shows that after 2 hours the temperature of the body is 85 degrees. The equation for the temperature contains a parameter that characterizes how fast the cooling goes. Use the measurement result to determine the value of this parameter.

What will the temperature of the body be 3 hours after it is placed into the room?

There will be a problem about second-order equations, on the material including their reduction to systems of first-order equations, linear and linear homogeneous second-order equations, the concept of linearly dependent and independent functions, fundamental set of solutions, etc.; examples are given below.

5a For the second-order differential equation

$$\frac{d^2y}{dt^2} = 2\frac{dy}{dt} + 3y: \qquad (*)$$

(a) Write a system of two first-order differential equations with two unknowns that is equivalent to the second-order equation $(*)$, taking $v = dy/dt$;

(b) Find the fundamental set of solutions of equation $(*)$, and write the general solution of this equation;

(c) Find the particular solution of equation $(*)$ satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$;

(d) For this particular solution, plot $\big(y(t),\, v(t) = dy/dt\big)$, $-\infty < t < \infty$, in the y-v-plane. Show the direction of the motion along the curve you drew.

5b The hyperbolic cosine and sine are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Are the functions $f_1(x) = (\cosh x)^2$, $f_2(x) = (\sinh x)^2$, $f_3(x) = 1$ linearly independent or linearly dependent?

Don't write simply **yes** or **no** as an answer: it must have some support.