MATH 2240-02 **Fall 2011**

Project 2: Model of an Epidemic

Look up Section 8.1 of the textbook.

We may model the evolution of an epidemic in a fixed population of N people (i. e., we don't take into account any births or deaths) by dividing the population into three different classes:

S = *Susceptibles*, those who have never had the illness and can catch it

I = *Infectives*, those who are infected and are contagious

R = *Recovered*, those who are immune to the illness, including those who had it and have recovered

Assume that the illness is mild (everyone eventually recovers), that the disease confers immunity on the recovered, and that the diseased are infective until they recover.

(a) Argue that the evolution of the epidemic may be modeled by the nonlinear system of first-order ordinary differential equations $S' = -aSI$, $I' = aSI - bI$, $R' = bI$, where the parameter b represents the reciprocal of the average duration of the infection, and a is equal to the average number n of people a person meets during a day (we measure time in days), divided by the total population N and multiplied by the probability p of getting the infection from meeting with an infected person. [Hint: Compare with models for predator-prey interaction.]

(b) Find the equilibrium points of the system of differential equations describing the evolution of the epidemic.

(c) Show that the onset of an epidemic can only occur if the susceptible population is large enough. Specifically, find the threshold value for S above which more people are infected each day than recover.

(d) Let $N = 1000$. Suppose that German measles last for four days on the average. Suppose that the typical susceptible person meets 25 people on the average each day and that the disease is transmitted in 1 of every 5 contacts with an infected person (the probability $p = 1/5$). Find the values of the parameters a and b. How small must the susceptible population be for this illness to fade away without becoming an epidemic? Verify by plotting the component graph for $I(t)$ for $I(0) = 10$ and for values of $S(0)$ that are twice smaller and twice larger than the threshold value found in part **(b)**. Plot over $0 \le t \le 30$ and discuss what you see.

Use the MATLAB routine `ODE45`, and explain how you did it, including the program (the `.m` file) and the command for solving initial-value problem for the system.

(e) Consider the illness we spoke about in **(c)**, with its average duration 4 days, $n = 25$, and $p = 0.2$. Suppose that the initial value of people immune to the disease is $R(0) = 0$ and the initial number of infected people $I(0) = 10$. Investigate how the spread of infection depends on the size N of the population. For what N the time at which the number $I(t)$ of infected people reaches its maximum is smaller, and for which N larger:

$N = 50$, or $N = 50,000$, or $N = 50,000,000$? After how many days this number reaches its maximum for $N = 50$, $N = 50,000$, $N = 50,000,000$?

Again use the routine `ODE45`, and explain how you did it.

(f) Using $a = 0.001$ and $b = 0.08$, find the expression for I as a function of S, $I(0)$, and $S(0)$, and plot I against S for various values of $S(0)$ and $I(0)$, $0 < S(0) \leq 1600$, $0 < I(0) \leq 1250$. Do it both using formulas and the MATLAB routine `pplane` (`pplane7` if you have Version 7 of MATLAB). Zooming in `pplane` near the origin, look at the long-time behavior. What are $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} I(t)$ equal to? Are these limits zero or positive? Interpret your graphs in terms of the model.

[HINT: Use the equation $\dfrac{dI}{dS} = -1 + \dfrac{b}{aS}$ that is obtained by dividing the equation $I' = aSI - bI$ by $S' = -aSI$.]

(g) Which of the equilibrium points of your system are stable? Which are unstable? Which are asymptotically stable?

Note that to answer these questions it's not enough to consider the linearized system as in Chapter 10 of the textbook.